DOCUMENT RESUME

ED 276 613 SE 047 610

AUTHOR LeFevre, Patricia
TITLE Exploring Fractions with Fourth Graders.
PUB DATE Apr 86
NOTE 96p.; Paper presented at the Annual Meeting of the American Educational Research Association (67th, San Francisco, CA, April 16-20, 1986).
PUB TYPE Reports - Research/Technical (143) -- Speeches/Conference Papers (150)

EDRS PRICE MF01/PC04 Plus Postage.
DESCRIPTORS Case Studies; *Cognitive Processes; Educational Research; *Elementary School Mathematics; *Fractions; Grade 4; Intermediate Grades; Interviews; *Mathematical Concepts; *Mathematics Instruction; *Symbols (Mathematics)
IDENTIFIERS *Mathematics Education Research

ABSTRACT

This exploratory study describes fourth-grade students' opinions and knowledge about fractions. A coherent and elaborate fraction knowledge base results when students establish connections between their conceptual knowledge and procedural knowledge of fractions. One possible juncture between concepts and procedures might involve establishing meaning for the formal symbols used in mathematics. Therefore, the meaning that students attach to the fraction symbol and their ideas about the utility of fractions were assessed through a written pretest and individualized interviews. The relationships of the fraction concept to what had been experienced and learned in the past and students' perceptions of the value in using various representational models of fractions were explored for the fractions 1/2, 1/3, and 3/4. Subjects were 15 fourth-grade girls from a private school. Seven research questions are reviewed with reference to two students, identified as novices, and an expert fraction student. Findings are discussed for their definitions of fraction, utility of fractions, comparisons of fraction and whole number, syntax and semantics of the fraction symbol, models of fractions, metacognitions, and the relationship of standardized achievement to performance. Appendices contain a four-page reference list, the interview questions and questions and responses, pretest with data, and observed performance and judged ranks. (MNS)

****
* Reproductions supplied by EDRS are the best that can be made *
* from the original document. *
****

ED276613

U.S. DEPARTMENT OF EDUCATION
Office of Educational Research and Improvement
EDUCATIONAL RESOURCES INFORMATION CENTER (ERIC)

☒ This document has been reproduced as received from the person or organization originating it.

☐ Minor changes have been made to improve reproduction quality.

• Points of view or opinions stated in this document do not necessarily represent official OERI position or policy.

"PERMISSION TO REPRODUCE THIS MATERIAL HAS BEEN GRANTED BY

Patricia Le Fevre

TO THE EDUCATIONAL RESOURCES INFORMATION CENTER (ERIC)."

EXPLORING FRACTIONS WITH FOURTH GRADERS

Patricia LeFevre

University of Delaware

Paper presented at the annual meeting of the American Educational Research association, April 20, 1986, San Francisco, California.

SE 047 610

BEST COPY AVAILABLE

EXPLORING FRACTIONS WITH FOURTH GRADERS

This exploratory study attempts to describe fourth grade students' opinions and knowledge about fractions and what it means to learn about them. From a constructivist perspective, knowledge is believed to be constructed by the learner and not merely transmitted from teacher to student in a passive fashion. Primary interest is in the mental and behavioral activities of students as they operate on mathematical concepts and materials of various representational types associated with fractions. A coherent and elaborate fraction knowledge base results when students establish connections between their conceptual knowledge (understanding, intuitions) and procedural knowledge (algorithms, routines) of fractions such that they support and enrich each other (Hiebert & LeFevre, in press). These connections will be made, it is hypothesized, when students have the ability, capacity and time to do so and they see them as useful and valuable. Alternatively, students may compartmentalize their conceptual and procedural knowledge so that they exist in isolation, each uninformed by the other. That is, formal procedures and concepts learned in school may not be connected to informal procedures and strategies for using fractions and intuitions about fractions gained through experience outside of school. This can lead to a fragmented, sterile, knowledge base, lacking generativity and making transfer of knowledge, beyond the particular context in which it was acquired, improbable. Problems that result from such an impoverished and inflexible knowledge system have been documented by recent studies such as the National Assessment for Educational Progress

(1983).

Research Design

Purpose

One possible juncture between concepts and procedures might involve establishing meaning for the formal symbols used in mathematics (VanEngen, 1949), or what Hiebert (1984) calls site one links between a symbol and its referent. That is, mathematical procedures taught and learned in school may be connected to students' intuitions and understanding about number when the component symbols have been embued with meaning through links to their concrete referents (representations of the ideas symbolized). When mathematical symbols are relatively meaningless to students it is not surprising that the routines and procedures using those symbols are likewise incomprehensible. These procedures must be learned by rote with no conceptual supports to monitor the procedure in progress or the answer that is produced. There does not exist a conceptual critic to signal when the sequence of steps is in error or when the result just does not make sense. The student has no way of judging the reasonableness of the answer or the steps used to get that answer. Their only recourse is to execute the procedural steps as best recalled; and recall may be difficult or erroneous. The student must rely soley upon surface features of the problem to prompt retrieval of the correct procedure. Steps of procedures that use the same symbols may be confused or grafted onto other procedures.

The situation should be very different for a student with elaborated knowledge of the concepts and principles that underlie symbols and symbol manipulation routines. While the procedures may run

automatically, errors should be readily detectable and possibly correctable. Conceptual knowledge makes available critics that signal when the actions taken or the result do not coincide with what the student understands or has experienced about number and numerical operations. Connections between formal symbols and their concrete instantiations enable students to represent a given problem in varied conceptually-sound ways. These representations can further aid the student in selecting appropriate procedures rather than retrieving any procedure that contains a particular symbolic configuration.

The purpose of this study was to explore the meaning that fourth grade students attach to the fraction symbol and their ideas about the utility of fractions. Students' understanding of fractions was assessed through a written pretest (see Appendix B) and individualized interviews (see Appendix A). The relationships of the fraction concept to what had been experienced and learned in the past and students' perceptions of the value in using various representational models of fraction (i.e. partitioned and shaded geometric area models, paper-folding, ribbon-folding, discrete set-subset models, and the number line) were explored for the fractions 1/2, 1/3 and 3/4.

Pretests were group administered (see Appendix B) prior to interviewing the students individually. The students were interviewed using one of two forms of the Fraction Interview (see Fraction Interview Forms A and B in Appendix A). There was an attempt to elicit the students' definition (understanding, concept) of fraction, their ideas about how fractions might be used and the helpfulness or difficulty of using the various representational models of fractions.

The clinical interview technique was especially valuable for ascertaining each student's concept of fraction and their flexibility in applying the fraction concept across concrete manipulables, pictures and symbols.

Research Questions

The specific research questions to be addressed by this study included:

1) What examples and definitions do students offer for fractions? (See Appendix A part A. 1.)

2) Do students vary in their perceptions of the utility of fractions:

a) ... in learning more mathematics?

b) ... in other school subjects?

c) ... outside of school? (See A part A. 2.-4.)

3) In what ways do students perceive fractions to be different from whole numbers? (See A part A. 5.)

a) Are fractions perceived to be easier, about the same, or more difficult to learn than whole numbers?

4) Do students demonstrate different levels of understanding in regard to the fraction symbol and vary in their understanding of the components of the symbol? (See A part B. 1.-2.)

a) To what extent are students bound to the form of the fraction symbol? Can they generate equivalent fractions; do they recognize that 1÷2 and 1/2 can represent the same amount?

5) Are students capable of generating models of the fraction concept embodied as:

a) geometric areas? (A part C. 1.)

b) discrete sets of objects? (A part C. 2.)

c) continuous linear models? (A part C. 3.)

6) Do students differ in their metacognitive awareness of:

a) possible misconceptions about fractions? (A part D. 1.)

b) the value of using concrete models of fractions? (A part D. 2.)

c) the different aspects of fractions captured by continuous versus discrete models of fraction? (A part D. 2.)

7) Do students of varied ability levels (assessed through CTBS and teacher recommendations) respond differently to the interview questions?

The research questions outlined above will be reviewed with reference to two students, termed a "novice" and an "expert" fraction student, using a case study format. These students represent opposite ends of a part-whole fractions competence continuum. The pretest data for both students appears in Appendix B and the interview data in Appendix C. The data for these students will be discussed and compared in an effort to uncover differences in the structure, coherence and generativity of their fraction knowledge bases, that may explain differences in their fraction competence.

Second, students' awareness of the difficulties associated with learning and using different representation forms (concrete manipulables, pictures and symbols) for fraction will be examined. In particular, the match between students' observed performance and their judgements about the helpfulness of the different models and physical

contexts will be examined across the fractions 1/2, 1/3 and 3/4. (See Appendix D.) An observed performance rank score was assigned to each of the five physical contexts (geometric areas, paper-folding, discrete sets, ribbon-folding, and number lines) based on the ratio and percent correct for that physical context. The context with the highest percent correct was assigned a rank of (1), the next highest a rank of (2), etc., with the lowest assigned a rank of (5). An exercise was included in the determination of percent correct only if it had been used for all fractions (i.e. 1/2, 1/3, 3/4). Hence some of the pretest questions have been omitted from this analysis. (Specifically these ommissions include the following pretest questions: 1) shading in 1/2 of the triangle, 2) shading in 1/3 of the hexagon, 3) shading in 3/4 of the square, 4) all parts of this question, 8) all parts of this question, 9) part (b), 10) part (b), 11) part (b).) Ranks for students' judgements of helpfulness were assigned based on their responses to the last interview question. Students' reports of which representational model (physical context) helped the most to understand fractions or was easiest to use was assigned a rank of (1). The representational model reported as least helpful or most difficult to use was assigned a rank of (5). The degree of difference between these observed and judged ranks was tested for all fifteen students using the Kolmogorov-Smirnov Test. Results and related observations are discussed.

Methodology

Subjects

The subjects in this study were 15 female fourth graders from a area private school. Five subjects, representing each of three ability

levels, were selected by their teachers to participate. All subjects were at or above the normal range of mathematical ability for their grade. The specific range in standardized achievement scores for the three groups were: a) 56-77%, above average; b) 83-93%, well above average; c) 97-99%, exceptional. The novice and expert fractions students discussed in this paper have achievement scores in the above average and well above average range as follows:

Subject 061	Total Math Score 77%	
(Novice)	Computations 91%	Concepts and Applications 46%
Subject 132	Total Math Score 93%	
(Expert)	Computations 91%	Concepts and Applications 92%

It should be noted that the computational scores for these students are identical, while the concepts and applications scores are very different. This data was not used to select the novice or expert fractions student, but will be used to aid in the interpretation of the data from this study.

Materials

A set of felt geometric shapes mounted on 8.5 x 11 inch cards, pipe cleaners, plastics disks (counters), and ribbon were available for students to manipulate and were featured in the interview questions. Paper and pencil were also available and a tape recorder recorded the proceedings.

Procedures

Prior to the interview, subjects were asked to solve fraction problems on a group administered pretest. Subjects were encouraged to answer every question on the test, even if they were unsure about a

correct response. They were advised that some of the problems might be unlike any they had seen before and that they should give their best guess as the answer. It was emphasized that the experimenter was most interested in what children think about fractions and less interested in whether they were correct. Therefore a response to each question would be most helpful, even if it was a guess. At a later date, subjects were interviewed individually according to the Fraction Interview format (Appendix A). Each was assigned a number for identification purposes to maintain confidentiality.

CASE STUDIES OF A NOVICE AND AN EXPERT FRACTION STUDENT

In this section of the paper, the seven research questions outlined above are reviewed with respect to the responses of a novice and an expert fraction student. First, the novice/expert distinction will be reviewed. Second, the method for selecting the novice and expert is explained. Next, the pretest and interview data for these subjects are examined to identify similarities and differences, as they address each of the seven research questions. Whenever possible speculations about the underlying conceptual model of the novice and expert are made. The objective is to determine whether differences in the competence of the novice and expert are attributable to overall differences in the amount of knowledge, a matter of accretion, or to a qualitative difference in the way that knowledge is structured, or both.

It could be that the novice just hasn't experienced or learned enough about fractions yet. On the otherhand, the system of fraction knowledge may be very different for the novice and expert. While the same

information may be present, the framework, hierarchical structure and networks of relationships may be more elaborate and/or coherent for the expert than for the novice. Perhaps there exist connections between the formal fractions concepts and procedures the expert has learned and her informal intuitions, understandings and experiences. These links might enable the expert to apply her knowledge to novel or difficult tasks and to transfer this knowledge beyond the particular context in which it was initially encountered.

The Novice Expert Distinction

Most definitions of expertise are domain specific, the expert being characterized as having extended knowledge of the domain which the novice lacks (Schoenfeld, 1983). Cognitive psychologists, using an information processing approach (Larkin, 1983a,b; Simon, 1978) describe expertise as the coordination of both automated procedures and "elaborated representations" (the subject's interpretation of a task in terms of underlying principles rather than some superficial features of a problem). These automated procedures and elaborated representations are combined for efficient problem solving. Students who construct a "basic" unelaborated representation of a given problem are doomed to an unconstrained and inefficient search for solution procedures, according to this view. If memory is pictured as a network where related pieces of information and procedures are linked to each other, then this approach would predict that a basic representation activates all portions of the memory network which are linked to elements of the given problem. All of this information would be retrieved whether it was appropriate or not.

Within the domain of mathematics, expertise would include the ability to represent a given problem in terms of mathmatical relationships and principles that would restrict the search for solution strategies and lead to more efficient retrieval and execution of procedures to obtain a solution (Larkin, 1983b). Students' performance on fraction tasks seems to fall short of the "expert" characterization in many cases, as demonstrated by the NAEP (1983) data. First, students seem to function at a very superficial level, depending on the exact duplication of contextual cues to prompt retrieval of procedures. The 1982 National Assessment of Educational Progress indicated, for example, that approximately half of the teenagers surveyed characterized mathematics as a matter of memorization. Students appear to be moderately successful in performing routine calculations for problems presented in standard form but are unable to transfer these skills to the solution of nonroutine problems that require analysis and comprehension, not merely the execution of memorized rules. The mathematical beliefs of many students can be described as instrumental (i.e. mathematics is a collection of methods for getting correct answers) rather than relational (i.e. mathematics is a system of relationships with a rational underlying structure), to use terms developed by Skemp (cited in Cobb, 1984). These divergent belief systems presumably lead to very different approaches to studying mathematics and probably very different conceptual systems.

A second example of students' lack of expertise in dealing with fractions are those instances where procedures, that have been learned or practiced less well, are inaccessible, are distorted, and/or are

confused with more familiar ones. For many students, there seems to exist an ever increasing set of context-specific rules for symbol manipulation. There does not exist a coherent and dynamic system of conceptual and procedural fraction knowledge where each informs and enriches the other. When these students are presented with a fraction problem, overlearned whole number counting procedures often intrude, producing errors (Behr, Post, & Wachsmuth, 1982; Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1981; Carpenter, Mathews, Lindquist, & Silver, 1984; Post, 1981; Post, Wachsmuth, Lesh, & Behr, 1985). These memorized rules are mechanically applied without regard for mathematical principles or the rationality of what results. They are especially susceptible to errors of three types. First, errors can occur when the surface clues contradict the mathematical relationships underlying the problem so that the wrong procedure is used. For example, presented with the problem "1/2 + 1/4", whole number procedures may be triggered so that the student adds the numerators and denominators. Second, surface clues may provide conflicting or insufficient information so that procedures are applied at random, with no basis for selecting among them. In the example above, the student may remember that there are times when you need to rename fractions in order to operate on them, but may forget when this is to be done. Third, the procedures which are retrieved may themselves be incorrect. This is highly probable when one must depend on episodic memory and the duplication of context-specific cues to prompt retrieval of procedures and there are no means to substantiate or confirm the correctness of the steps to solution or the answer that results.

Researchers who analyze student errors in mathematics have documented the systematic, rule-governed, nature of these errors. For example, Brown and Burton (1978), and VanLehn (1983) have outlined children's procedural "bugs", incomplete and/or incorrect procedures, in multidigit subtraction. These bugs were not taught to students, but were constructed by them in the midst of their attempting a solution, although children may indeed maintain that they were taught that way. Another illustration is Erlwanger's (1975) account of one student's elaborate and incorrect procedures for getting answers to computational problems. This student, though he was perceived to be a good student, was using faulty symbol manipulation routines and building an ever increasing storehouse of incorrect solution methods for producing answers.

Tactical versus Strategic Knowledge. In a discussion of expertise, Schoenfeld (1983) distinguishes between tactical and strategic decisions in problem solving. Tactical decisions involve the local selection of problem solving procedures, as outlined above. Strategic decisions involve executive or managerial decisions such as the allocation of resources. Schoenfeld suggests that strategic decisions are most important for the success of a "nonproficient" expert in problem solving. Beyond choosing a path to solution, strategic knowledge is used to assess progress at the local and global level so that tactical failures call for a strategic review. Schoenfeld further notes that novices often lack these "monitors" for review. Furthermore, experts often salvage aspects of their flawed approaches so that they profit from an unsuccessful trial. Novices, on the otherhand, may scrap a

procedure in toto and begin the solution process again from scratch, uninformed by experience.

The Novice and Expert Fraction Student

The expert fraction student may be viewed as a nonproficient expert in the domain of fractions. Hence, Schoenfeld's concern about strategic as well as tactical knowledge may be applicable. In reviewing the data of these students and selecting a novice and expert, consideration was given to the strategic and tactical knowledge applied by them. In particular, attention was focused on: a) the tactics used to generate solutions (e.g. elaborated versus basic problem representations); b) the use of strategic knowledge (i.e. monitors for review and salvaging from unsuccessful attempts); c) differences and similarities in the structure of their fraction knowledge (evidence for a relational rather than an instrumental belief system and a coherent, integrated, system of fraction concepts and procedures rather than a fragmented one. These aspects of expertise will be considered for the identified novice and expert fraction students in the discussion of the research questions identified above.

Novice and Expert Selection. Examination of the overall performance of all students across the different fractions (i.e. 1/2, 1/3, 3/4) and representational types revealed a variability in the rate of success (see Appendix D). Correct responses to a total of 54 problems, used with all fractions, ranged from 17 correct out of 54 problems, for subject 061, to 47 correct out of 54 problems, for subject 132. Based solely upon correctness of response, subject 061 demonstrates the weakest performance and subject 132 the strongest

performance. The median and mode success rate was 30 out of 54 or 56% correct (a total score for 5 subjects). While these data are informative, more evidence for distinctions in expertise was needed to address the issues of differences in tactical and/or strategic knowledge and cognitive structure.

When students' responses to the research questions were examined, marked differences were discovered in the ways that subject 061 and subject 132 defined fractions, the meaning they attached to the fraction symbol, their perceptions of the utility of fractions and their awareness of the difficulty or helpfulness associated with using the different representational models. A stable part-whole concept of fraction enabled subject 132 to successfully apply her fraction knowledge to novel problems and varying contexts. For example, experience gained in one form (partitioning a circle into fourths) to be used in the representation and solution of a problem in a different form (partitioning sets of objects into fourths). Conceptual knowledge also guided the solution process by signalling errors in progress or in the final product.

In contrast, subject 061 had a fragmented system of fraction knowledge. She fluctuated between a part-whole and part-part interpretation of fraction and this lead to confusion regarding the fraction symbol and its components. Different contexts prompted the retrieval of different interpretations and applications of the part-part concept were incorrectly and inconsistently applied. Subject 061 recognized this confusion. On the basis of this evidence subject 061 was chosen as the novice and subject 132 was chosen as the expert

fraction student in the hope of explicating how differences in their knowledge (amount or structure) might result in these large performance differences. (See Appendix C for the Fraction Interview data of subject 061 and subject 132.) Each of the research questions are reviewed below with reference to the novice and expert students' responses.

Findings

Definition of Fraction

The first research question concerned the definition and/or example that students offered for a fraction. Responses could indicate the fraction interpretation used by students (part-whole or part-part) and connections between that interpretation and the fraction symbol. Subject 061 gave neither a definition nor an example of a fraction. When the interviewer asked whether sharing a pie or pizza might be an example, subject 061 responded, "yes".

Excerpt 1:

Interviewer: ...any idea why those things are examples of fractions?

Subject 061: ..oh, because you cut them.

I: Okay, and what does the cutting have to do with it?

061: Well, you're cutting into sections.

I: And what is a fraction in regard to those sections? I mean, how does a fraction match up with the sections? Any idea?

061: (no response)

As this dialogue illustrates, when a real-world context for using fractions was provided, subject 061 acknowledged it as an example and

seemed to indicate a partitioning (cutting, sectioning) notion of fraction. However, it was not clear that this cutting notion was truly associated with fractions. Perhaps subject 061 was adept at reading social cues and believed that teachers and interviewers ask questions to provide information rather than to extract information and learn something new. In that case, she would assume that an example had been provided by the interviewer and the dialogue follows. Since further probes yielded no response, this hypothesis cannot be rejected.

In contrast to subject 061, subject 132, the expert, gave a precise part-whole definition of fraction for which she provided conceptual understanding. She did not seem to be parroting back a memorized statement.

Excerpt 2:

132: A fraction is a part of a whole.

I: Can you give me an example?

132: Like one half is half of a whole?

I: What does it mean to say it's half of a whole?

132: It's split into two even parts.

I: Okay. And what does even mean?

132: equally.

Subject 132 defined fractions using a part-whole scheme. That is one interpretation of rational number as the partitioning of continuous quantities or discrete sets into equal-sized subparts or subsets (Behr, Lesh, Post, & Silver, 1983; Kieren, 1976). Typically, first notions of

fractions are inclusive (part-whole) rather than exclusive (part-part) and refer to sharing operations (Vergnaud,1983). Subject 132 indicated that parts must be equal, evidence for a part-whole scheme that has applications to other topics in mathematics as well. (See Riley, Greeno, & Heller (1983) for addition and subtraction applications and Greeno (1983) for applications to geometry.)

Subject 061 revealed only the possibility of an association between partitioning actions and the term "fraction". Because of her limited response one cannot discern the level of her partitioning or fraction concept. These will become clearer as the interview progresses and partitioning actions are observed. The relational character (part-whole structure) of subject 132's definition and example was absent from the restricted response of subject 061.

Utility of Fractions

The second research question addressed issues related to the perceived utility of fractions. In most cases, the responses of the novice, subject 061, and the expert, subject 132, were similar. Subject 061 agreed it would help you learn about math if you know about fractions. Again this merely affirms the interviewer's question. Subject 132 saw fractions as..."something kinda important in math." When probed about this she responded, "Because, when you get older probably they'll have an uneven division problem and you'll have to say, like, 1/2 or 1/3 of this or that." Subject 132's ability to apply this knowledge was demonstrated later in the context of partitioning discrete sets in the presence of inconsistent cues (e.g. finding 1/2 of a set of 5 or finding 1/3 of a set of 7). Both subjects thought that fractions

would be studied this year or next year and even into high school or college. At that time, however, fractions would look different. The novice, subject 061, conjectured there would be bigger numbers, 1000 or 1,000,000, in the fractions, while the expert, subject 132, suggested they would be harder..."something like 23/25 ÷ 43/55." Increased difficulty for the novice meant that fractions would contain larger numbers; for the expert it meant that fractions would be used in more complex algorthims. Science was mentioned by both subjects as a nonmathematical context in which fractions could be used.

The most revealing question in this section of the interview asked students to think of a story that might have a fraction in it, where fractions might be used, so that they could help a friend to understand what a fraction is. In the context of this question, students sometimes revealed facets of their understanding of fraction that had not been previously verbalized. Unfortunately, subject 061, the novice, gave no response to this question, as did three other subjects. Such information just wasn't available to her, or she chose not to volunteer it.

Subject 132's story was very elaborate.

Excerpt 3:

I: ...Can you think of a story where fractions might be used?

132: Once upon a time there was a little girl in America. And she went to the supermarket...and they were selling eggs at 36¢ a dozen. And she only had enough money to buy 6 eggs and the money that she paid for that would be 18¢-half of the 36¢,

because 6 is half of a dozen.

Subject 132 indicated that this story was taken from one that she had encountered in a math meet. (Apparently, subject 132 is a member of a group of students in the Delaware Mathematics League that holds competitive mathematical problem solving meets between schools, with a team from each grade level represented.) She further indicated that she had not solved the problem correctly the first time, but when a similar problem came up later she "got that right".

This fraction story contains elements of a ratio interpretation of rational number as a comparative index, or the notion of constancy of relationship between quantities that are not equivalent (Behr et al., 1983).

> If a dozen (12) eggs costs 36 ¢, then 6 eggs cost 18 ¢.
> 1/2 of 12 eggs is 6 eggs and 1/2 of 36 ¢ is 18 ¢.

The relationship between eggs and cost is seen to be constant, if you take half of each respectively.

Research questions one and two demonstrated that subject 132 has a more elaborate, coherent and accessible system of fraction knowledge, whereas subject 061 demonstrated, at best, a limited partitioning idea of fraction as a cutting or sectioning operation. Evidence for this notion was provided by this excerpt:

Excerpt 4:

I: Can you think of any time when you might use fractions?

061: ...playing a game...parchessi...I lost most of the pieces and we only have a couple, so we have to divide them equally.

This was the first indication that subject 061 could apply a partitive notion of fraction where equal shares are important. It is significant that subject 061 related this idea to a concrete real-world problem she had solved by partitioning a set of game pieces into equal-sized subsets. These types of partitions may be accomplished without using a part-whole anticipatory scheme however. (See Hiebert & Tonnessen, 1978; Pothier & Sawada, 1983.) Later manipulative questions show that subject 061 can partition sets of 2, 10, and even 5 (inconsistent cue) into halves and can show three-fourths of a set of four objects. Though she had noted the condition of equality of parts, her ability to apply this knowledge to sets seemed to be limited to a halving mechanism, with three-fourths of four being one exception. She was unsuccessful in partitioning sets into thirds.

Comparisons of Fraction and Whole Number

Research question three was aimed at discerning students' perceptions of the differences and similarities of fractions and whole numbers. Both the novice and expert perceived fractions to be lower, smaller, than whole numbers such as 15 or 125. Fractions were necessarily proper fractions, not inconsistent with a part-whole interpretation. Subject 132 also noted that fractions were..."different the way they're written." A question about similarity prompted no answer from subject 061 and a part-whole response from subject 132, " They're kinda the same because

if you say two halves that's equal to a whole." Fractions were reported, by both subjects, to be more difficult to learn about than other numbers.

Syntax and Semantics of the Fraction Symbol

The syntax (form) and semantics (meaning) of the fraction symbol were the focus of research question four. When asked to write a fraction, subject 061 wrote "4/3", whereas subject 132 wrote "125/126". Subject 061 was the only subject to write an improper fraction. To the question, "Is there another way to write the same amount as 1/2?", subject 061 answered, "not sure". Subject 132 gave an equivalent fraction as her answer, "3/6", demonstrating her ability to generate equivalent fractions. The probe, "Does 1/2 = 2/1?" (1/2 = 2/1 was written out on paper for the subjects to inspect) yielded very different responses. Subject 061 said they were the same. Perhaps the fraction she had written, 4/3, could also be written as 3/4 in her view. The positions of numerator and denominator seemed to be interchangeable for this subject. Indeed, subsequent dialogue indicated that she did not know why there were two numbers in the symbol nor what the numerator (top number) or denominator (bottom number) in the symbol mean.

Subject 061 appeared to lack site 1 connections between form and understanding for fractions (Hiebert, 1984). It seemed that she had not established concrete referents for the fraction symbol. Alternatively, subject 061 may have been confused and unable to choose between conflicting referents for the fraction symbol. On the pretest both part-whole and incorrect part-part responses had been given. When asked to write the fraction that matched the shaded parts of four shapes (see

Appendix B, question 4.), subject 061 used a part-part strategy. The number of unshaded parts was recorded as the numerator and the number of shaded parts as the denominator. Later the correspondence between numerator and denominator and shaded and unshaded parts was reversed. These inconsistencies within one interpretation (part-part) and fluctuations between interpretations (part-part versus part-whole) illustrated the instability of the fraction concept and its relationship to the fraction symbol. Confusions between part-whole fraction relations and part-part ratio relations are quite common in children's primitive rational number thinking according to Lesh, Landau and Hamilton (1983). On the otherhand, subject 061 may have been at the bottom of a U-shaped learning curve (Strauss & Stavy, 1982), in that she recognized that different interpretations of fraction existed but was unclear about when each should be invoked. Perhaps a once stable part-whole concept of fraction was now ambiguous because a new part-part interpretation was possible. In contrast subject 132 uniformly applied a part-whole interpretation of fractions to all fraction problems. There did not exist alternative meanings; no choice was required.

Subject 132's response to the probe follows:

Excerpt 5:

I: Does 1/2 = 2/1?

132: No, because 2/1 is equal to two wholes and 1/2 is equal to one half of one whole.

I: What would it mean if you wrote just one of the numbers? (examples like 1/ and /2 were written on a piece

of paper for the subjects inspection)

132: No. If you left out the numerator then you wouldn't know how many of the denominator you had and if you left out the denominator the numerator would mean nothing, 'cause it could have been 1/16 or 1/8 or 1/4 or something like that.

I: What do the top and bottom numbers in the symbol mean?

132: (bottom)...it means that the whole is divided into 126 parts. (top)...125 of the 126 parts.

The novice and expert both stated that the bar (slash, line) in the fraction separates or divides the two numbers. There was no indication of the indicated quotient interpretation of rational number.

For subject 132, then, the fraction symbol was connected to a part-whole scheme that was well established. The components of the symbol matched components of this scheme so that the denominator connoted the total number of equal parts and the numerator connoted the number of parts under consideration. In contrast, the fraction symbol and its components seemed to convey little meaning or the meaning was unclear for subject 061.

When asked if another symbol could be used in place of the bar, slash, this dialogue followed:

Excerpt 6:

I: ...is 1/2 like 1 ÷ 2? (1/2 and 1 ÷ 2 were written on a piece of paper for inspection)

061: maybe

I: Like this (pointing to the illustration), would 1/2, if you wrote it 1 ÷ 2, would they mean the same thing?

061: No

Excerpt 7:

I: Could another symbol be used instead of the bar, for example, if someone wrote 1/2 and someone else wrote 1 ÷ 2, would they mean the same thing?

132: Kind of, because you'd have to split up the one into halves or fourths- Oh no, it won't mean the same thing, because you have to split up one into fourths to make it go into two.

It appears that subject 132 interpreted this question as a series of operations (take 1/2 and then divide by 2), rather than a comparison of 1/2 and 1 ÷ 2. Both subjects essentially responded no when asked if 1/2 and 1 ÷ 2 mean the same thing. Perhaps the use of the term mean was the problem, especially since neither subject seemed to attach meaning to the bar/slash in the fraction symbol. A second question regarding the value of 1/2 and 1 ÷ 2 may have yielded different results.

Models of Fractions

The next set of related questions were designed to asess students' abilities to generate concrete models of the fractions 1/2, 1/3, and 3/4 across different physical contexts or embodiments. These included geometric areas (i.e. a rectangle (square), circle, parallelogram (rhombus), and regular hexagon), paper-folding, discrete sets, and

ribbon-folding. The performance in each of these contexts for subjects 061 and 132 are compared in reference to the use of tactical and/or strategic knowledge and the cognitve structure they reveal.

Geometric Areas. In this portion of the interview, subjects were presented with felt geometric shapes mounted on 8.5X11 inch cards and asked to partition these shapes to show the fractions 1/2, 1/3, 3/4, using the pipe cleaners provided. These shapes included a square, circle, rhombus, and regular hexagon. Since this was an exploratory study, two different presentation orders were used. Eight subjects were asked to show 1/2 of the square, circle, rhombus, hexagon, then 1/3 of each shape, etc.. Subject 132 was one of these eight subjects. Seven subjects were asked to show 1/2, 1/3, 3/4 of the square, then 1/2, 1/3, 3/4 of the circle, etc.. Subject 061 was one of these seven subjects. Presentation order did not seem to effect performance except that the latter presentation order, partitioning each shape into halves, thirds, fourths before progressing to the next shape, resulted in a more rapid pace. (The figures did not have to be changed for each problem.)

When the subject indicated that they had finished a partition, the interviewer quickly recorded her response by drawing a line to represent pipe cleaner placements on the respective geometric shapes. These data appear in Appendix C for subjects 061 and 132 with each fraction response shown separately (1/2, 1/3, 3/4) and in Figure 1 where the entire set of area responses are displayed for these subjects. The subjects' performance on area pretest questions will also be discussed to obtain the fullest possible picture of each subject's capabilities. (See Appendix B for the pretest data on subjects 061 and 132.) The

partitioning actions of the novice and expert will be examined for each fraction separately, across the different geometric shapes. This order agrees with the presentation order used for subject 132 but not for subject 061, who partitioned each shape into halves, thirds, and fourths, indicating three-fourths, before proceeding to the next shape. On the pretest, a loose criterion was used to evaluate the partitioning and shading responses. Subjects were credited with success unless they clearly mispartitioned or incorrectly shaded a figure.

Partitions to show 1/2 were successfully completed by the novice and expert on both the pretest and interview questions, with one exception. Subject 061 placed the pipe cleaners on the square so that it was partitioned into fourths and pointed to one-fourth when asked to indicate one-half. This same quartering strategy was later used successfully with the hexagon however. In this case, subject 061 pointed to two of the fourths to correctly show 1/2. Based on these data, it is concluded that the novice and expert were both capable of using a halving mechanism correctly.

When asked to partition the geometric shapes into thirds and fourths, subject 061 gave no response, even though fourths had been used in two instances to show 1/2. Because of the presentation order used here, subject 061 did partition the shapes into halves, interspersed with two occasions of no response for each. The alternative presentation order would have highlighted this lack of respose to a greater extent.

Examination of the pretest data for subject 061 (Appendix B) show that she did correctly partition the rectangle and parallelogram, but

shaded 2/3 rather than 1/3 of each. Incorrect vertical cuts were used to show 1/3 of the circle and hexagon, which were also incorrectly shaded. Based on this performance, subject 061 appeared to be in transition between two levels of partitioning capability described by Pothier and Sawada (1983) as Level I, sharing, and Level II, algorithmic halving.

Pothier and Sawada outline four levels of partitioning competence that are supported by their data. Each level is characterized by (a) the development of a construct or key idea, (b) a routine or algorithm used to make partitions, and (c) a particular domain of application for partitioning capabilities. Level I involves: (a) the key concepts of breaking, sharing, halving; (b) an allocating pieces algorithm; and (c) a domain that extends to social settings and counting numbers. The sharing example of parchessi game pieces illustrates this partitioning level for subject 061. At Level II, algorithmic halving, students use: (a) the construct of partitioning into halves with no notion of equality; (b) a repeated dichotomies algorithm; and (c) a domain that extends to 1/2 and other unit fractions whose denominators are powers of two. Since subject 061 demonstrated the capability of modeling a fourth (1/4), it is hypothesized that she was in transition between Level I and Level II.

When students are at Level III, evenness, (a) equality and congruence are noted and halving is meaningful, (b) halving and geometric transformations are used (sliding, rotating partitions), and (c) the domain of application extends to all unit fractions with even denominators. Level IV, is characterized by: (a) the construct "even"

and use of a new first move free from dominance of the dichotomizing move, and use of geometric transformations; (b) an algorithm that uses exploratory measures such as trial and error and counting; and (c) a domain that extends to all unit fractions.

Unlike the novice, the expert, subject 132, was capable of partitioning at least at Level IV and perhaps beyond. She was able to partition all geometric shapes to show 1/2, 1/3, and 3/4 and was the only subject to successfully show 1/3 of a hexagon. The use of radial partitions to alternating vertices of the hexagon illustrated her Level IV proficiency. Subject 132 consistently used a part-whole fraction relation to name the fraction that matched the shaded parts of the geometric shapes. (See Appendix B, question 4.) In fact, for the hexagon she offered two correct responses 1/2 and 3/6.

Paper-folding and Ribbon-folding. Subjects were asked to fold continuous materials, both area and linear, to show the fractions 1/2, 1/3, and 3/4. In the paper-folding (area) questions subject 061 correctly partitioned and labeled 5 3/8 X 11 inch pieces of paper into halves and thirds. She was unsuccessful in generating a model for fourths after two attempts. While the paper was folded into four sections, they were unequal. A fold-up strategy was used. Starting at one end of the paper, the subject decided where the first fold, designating one-fourth, would be then folded that section over and replicated the fold twice. Because the first fold was inaccurate, the last section was left unequal to the first three sections. A second attempt was not successful either; the subject indicated that the sections should be even, but apparently was unable to produce them.

It is interesting that subject 061 was able to produce a paper-folding model of thirds though she would not attempt to partition geometric shapes into thirds. Prehaps the presence of materials to be manipulated (paper to be folded) was a more compelling context than the mounted shapes. The subject may have felt a stronger obligation to respond here. On the otherhand, she may simply have felt more comfortable with the interview situation at this point and less apprehensive about offering a response. Some of the students volunteered that they had used paper-folding techniques in art class. It is possible that subject 061 had established associations between "thirds" and this particular folding strategy. It is not clear from the interview what explanation is appropriate.

Later incorrect responses to ribbon-folding exercises show that procedures for folding into thirds are not stable for subject 061 across linear and area models. Subjects were asked to fold a ribbon to show halves, thirds, and fourths and to cut and to give the interviewer a one-half, one-third, and three-fourths piece, respectively. These were taped to pieces of paper and labeled by the interviewer. Given this task, subject 061 used three folds for thirds, producing four sections, and four folds for fourths, producing five sections. There was a confusion between the number of parts needed and the number of folds necessary to produce those parts. Similar confusions for subdivisions of area are described for young children (about age four) by Piaget, Inhelder and Szeminska (1960).

The performance of the expert fraction student, subject 132, was in marked contrast to that of the novice. Subject 132 was successful in

producing paper-folding and ribbon-folding models for all fractions (1/2, 1/3, 3/4). The strategies used were efficient and applied without hesitation. For example, fourths were produced by successive dichotomies. The paper (ribbon) was folded into halves and halved again to readily yield four equal-sized sections. Thirds were more difficult, but adjustments in the folds easily gave the correct result.

Linear Models. Research has shown that linear, number line, models are difficult for students to use (Behr et al., 1983; Larson, 1980; Leinhardt, 1983). The ribbon-folding exercises used here were similar to the first type of pretest number line problem in which subjects were presented with a line segment from 0 to 1 and were asked to indicate where 1/2, 1/3, 3/4 were located. The given line segment, like the given length of ribbon, represented the unit to be partitioned. The subject was not required to determine what the unit was before proceeding; it was understood to be the whole as given. This problem involved what Behr et al. call consistent incomplete cues that require the subject to add features (partitions) to complete the model. Problems arise and difficulty increases when line segments that extend beyond 1 are used, as they are on the pretest. Now the subject must first determine the unit, before designating the fraction locations. The use of irrelevant (neutral, extraneous information such as partitioning marks for fourths when 1/2 is to be located) and inconsistent cues (conflicting information) add further to the difficulty.

It is not surprising that the novice fraction student was unsuccessful for most number line problems. Curiously, she did

correctly locate 1/2 in the most difficult problem context (see Appendix B, question 9, part e.). In this problem, there were inconsistent cues with a segment from 0 to 2. Success here was inconsistent with failures on much less difficult problems. Perhaps it is a fluke.

Number lines proved to be difficult for the expert as well. Those number lines with consistent incomplete cues (9 a. b., 10 a. b., 11 a. b.) presented no problems. The introduction of line segments from 0 to 2 led to common errors, indicating that the subject incorrectly assumed the entire segment represented the unit to be partitioned. The data for expert and novice alike reaffirm what researchers have shown; number line models are not easily used by students to represent fractions particularly when line segments beyond the unit from 0 to 1 are pictured.

Discrete Set Models. The last context to be discussed employed discrete set models. Subjects were asked to show 1/2, 1/3, 3/4 of various sets. The first problem, for each fraction in the interview, contained consistent complete cues (e.g. 1/2 of a set of 2 disks). The second problem used consistent complete cues but the cardinal number of the set was a multiple of the denominator (e.g. 1/2 of a set of 10 disks). The last problem employed inconsistent cues so that the set size was not a multiple of the denominator (e.g. 1/2 of a set of 5). This last problem was very difficult and required some problem solving. Strategies used and verbalizations here were most informative.

The novice student was able to show 1/2 for sets of 2, 10, and even 5 disks. Beyond that, she offered no response for thirds and questions about fourths yielded this dialogue:

Excerpt 8:

I: 3/4 of 4?

061: 3

I: 3/4 of 12?

061: don't know

I: 3/4 of 6?

061: (long pause)

I: How would you start to do it? (pause) No idea?

Subject 061 moved 4 disks away and indicated 3 out of four as a response.

I: Okay, so you'd, 3 out of how many, first?

061: 4

I: Okay, so you'd start off with that and then, what do you have left?

061: 3

I: So, you have 3 out of 4 and how many are left?

061: 3

I: Are you sure?

061: 2

I: Right, so there's 2 left. What are you going to do with those 2 so that you can show three-fourths of that? (pause) What do you think? (pause) Can you think of any way of doing that one?

061: No

The strategy that subject 061 chose here required generating and

maintaining the answer to subgoals so that this information could be combined to arrive at a final solution. The amount of memory used to store the subgoal information may have been so overwhelming that subject 061 had a limited amount of cognitive resources left for operating on the information to get an answer. She also lost information or retrieved incorrect information (e.g. answering that there were 3 rather than 2 left when 4 were taken away from 6). Recording the answers to subgoal problems enroute to the final answer would have eased the processing load and freed up memory space for problem solving. However, none of the fifteen subjects used the paper and pencil available to record answers. It could be argued that the disks themselves carry this information. A set of six could be separated into subsets of four and two each. Then, the subsets could be partitioned to show three-fourths (i.e. 3/4 of 4 = 3; 3/4 of 2 = 1 1/2) and added to get the answer 4 1/2. There are many places in this solution strategy where confusions and errors could be made. Subject 061 appeared unable to get beyond the first few moves, partitioning the set of six into subsets of four and two and finding 3/4 of 4. (One subject, 072, successfully completed this sequence of moves to produce the answer 4 1/2.)

Pretest data for subject 061 (see Appendix B, questions -8) in many ways match the interview responses, but, differences do exist. A correct answer was produced for 1/2 of a set only for a set of two, while in the interview correct responses were given even for the most difficult problem (1/2 of 5). For sets of six and nine circles, only one circle was shaded to show 1/2. As had occurred in the interview, the novice was unable to show 1/3 of any set (3, 12, or 10). Correct

shadings were performed for 3/4 of 4 and 3/4 of 12, however. It is interesting that the subject responded, "don't know", when the same question was posed, 3/4 of 12, in the interview.

As in the geometric context, part-part responses were given when the subject was asked to write the fraction that matched the shaded part of each set. Now, however, the shaded circles were recorded as the numerator and the unshaded circles as the denominator. The reverse of the form previously used. Decisions regarding what information should be recorded as numerator or denominator seemed quite arbitrary for subject 061. Once a form was chosen, however, it is used consistently within a problem set, much the way a "patch" is used in VanLehn's (1983) Repair Theory. The fraction symbol, at least for this novice, seemed to serve as a recording device for each part in the part-part relationship. Correspondence between positions in the fraction and shaded or unshaded parts was assigned, it seemed, at random by the subject.
Since a part-part interpretation was inappropriate, there were no clues regarding a correct correspondence here. Such actions resemble what VanLehn describes as "bug migration" (shifting from one systematic error to another over a short period of time).

Unlike the novice, the expert fraction student correctly responded to all pretest and interview discrete set problems, with two exceptions. In showing 1/3 of 10 circles, she shaded in 2 1/2 circles, and there was a reversal error (6/4 versus 4/6) for one part of question 8 (see Appendix B, question 6 c. and 8.) A most revealing interview response was given for the problem 3/4 of 6:

Excerpt 9:

I: What would 3/4 of 6 (disks) be?

132: (long pause) Wait a minute, I'm not sure about this one.

I: Take your time. (pause) How do you think aout that one?

132: I'm trying to think...(in soft pensive voice)

I: Any id...(subject 061 breaks in)

132: If I divide one into...divide one into, one fraction and then try to pick out how many would make... (pause)

I: What could you do first to help you? (pause) You want to find 3/4 of 6. (restating the problem very softly, as subject 132 seems to be in deep thought)

132: I could try dividing one little circle here into um... (long pause)

I: Suppose you could move them around, would that help?

132: Let's try.

After looking at the six disks, subject 132 arranged them carefully into a circular configuration and announced:

132: Um, four and a half (4 1/2).

I: How did you think about that to begin with?

132: First I thought, how am I going to get 3/4 out of 6? And then I split some pieces up into like fourths and that didn't work.

I: That was a lot to keep track of wasn't it.

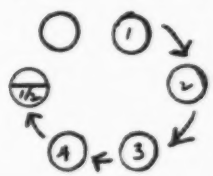
132: uhuh, thirds and that didn't work. And then you told me to move them around and then that worked.

The subject took a suggestion as a cue for finding a solution strategy.

I: Okay. How did that help you when you moved them around?

132: Because, when I moved them in a circular shape, I said, 1,2,3,4, and then I found that I had to split some things in half.

For this difficult problem, the expert displayed the most complete evidence of using tactical and strategic knowledge, as well as a dynamic system of fraction knowledge that was stable and transfered across various physical contexts and fractions. Monitors for review (Schoenfeld, 1983) came into play when the expert noted that splitting the disks mentally into fourths would not work. However, the idea of splitting the disks was then tried for thirds, evidence for salvaging from a previous attempt. This also did not work. At this point, the subject read the interviewer's question as a cue. Other subjects had been given the same suggestion about moving the disks and three had responded in a manner similar to the subgoal generation strategy used by subject 061. Unlike these students, subject 132 used her knowledge of circular partitioning to configure the disks and directly read the answer from the configuration, as illustrated below:



This strategy had a much lower cognitive demand in that subgoal

information did not have to be stored and combined to generate the response. The answer followed directly from the representation used.

This episode demonstrated that subject 132 had "critics" available to evaluate performance as she progressed and to call for review when movement toward the goal was not apparent. It is not clear from subject 132's response why the first strategy for partitioning six disks into fourths was aborted, other than her statement it didn't work. Perhaps, she recognized that the cognitive load associated with it and her inability to retain all the necessary information. Or, she may not have recognized how the strategy could lead to a solution. For whatever reason, the fourths and thirds (incorrect) strategies were terminated.

As Lesh et al. (1983) note, "when we say a student understands fractions, we mean, in part, that he or she can express fraction ideas presented with circular regions using rectangular regions, or using written symbols" (p. 270). For subject 132, understanding fractions was evidenced by her ability to translate among several representational modes (i.e. geometric areas, paper-folding, ribbon-folding, discrete sets, and formal written symbols). Not only was she able to translate between the fraction symbol and the various manipulative and pictorial representations and vice versa, but, translations between different models became powerful and efficient problem solving tools (e.g. when a circular structure was used to partition a set of six disks). While subject 132's knowledge of fraction was highly elaborated, well connected and accessible for problem solving it was limited to a single part-whole interpretation of fraction. She demonstrated how far a single coneptual interpretation of fraction can take a student when that

interpretation is sufficiently enriched and flexibly applied. Such evidence does not exist for the novice, subject 061.

Metacognitions

The sixth research area pursued in this study concerned students' metacognitive awareness of misconceptions, incorrect ideas about fractions and the fraction symbol, and the advantage or difficulty in using different models of fraction. When questioned about the kind of mistakes a confused student might make, subject 061 responded, "I am confused." She was aware that fractions were difficult for her and the data support this view. The expert fraction student mentioned reversals of numerators and denominators as typical errors. She too was aware that some fractions problems were very difficult and required extra effort, as in finding 3/4 of 6.

When asked, "Which helps the most (least) in trying to understand what a fraction is?" or "Which is the easiest (most difficult) to use?", both the novice and expert stated that area models were most helpful, easiest to use. Number lines were perceived to be least helpful , most difficult to use. The question of match between these judgements and observed performance was of interest. For the expert, there existed total agreement between judgements and observed performance. (See Appendix D for rankings of observed performance and judgements per subject.) For the novice, there was agreement only between the judgement and observed performance for number lines. Though area models had been judged as most helpful, they ranked one level above number lines based on performance. Performance in ribbon-folding, discrete sets and paper-folding exceeded that in the area context.

Both subjects were aware of the difficulty in learning about fractions compared to whole numbers. It is not surprising that the judged and observed performance ranks agreed for the expert but not the novice student. All students who responded to the question of difficulty (13 of 15) recognized that number lines were very difficult to use. The expert student furthermore was equally adept at using the other interpretations, so they all received a rank of (1) and her judgement would agree in any case. The novice however had difficulty showing fractions beyond 1/2 using area or set models and had limited success in any mode. The judgement was just more difficult for the novice than the expert.

Examination of judged versus observed performance ranks for all students was conducted. There was perfect agreement between students' statements that number lines were the most difficult, least helpful, model to use. For the most helpful, easiest to use, there existed agreement between the judged and observed performance of seven students, while disagreement to a greater or lesser degree existed for eight students. As pointed out above, this judgement was more problematic than the latter judgement concerning number lines. The Kolmogorov-Smirnov one-sample test was used to test the goodness of fit between judged and observed ranks where the distribution for judged ranks was a rank of (1) for each student. (See Figure 2 in Appendix D.) Since the significance level ($p<0.01$) associated with $D\geq 0.53$ was less than $\alpha = 0.01$, the null hypothesis that no difference in rank distribution existed was rejected in favor of H_1. There existed a statistically significant difference between subjects' judged and observed rank

performance.

It should be further noted that geometric areas were most often judged to be helpful, easiest to use (11 of 15 responses). Three subjects chose paper-folding and one chose ribbon-folding. Discrete sets were never judged to be most or least helpful, easiest or most difficult to use. This makes sense when the observed performance rank for this category is examined. Performance shows that discrete sets never received a rank of (1) or (5) based on percent correct. Therefore subjects judgements are suitable here.

Relationship of Standardized Achievement to Performance

The last research question concerned the relationship between student performance in this study and their reported standardized achievement. Subjects in this study represented three above average ability groups: (a) exceptional, (b) well-above average, and (c) above average. The expert student for this study came from the well-above average group (not the exceptional group), while the novice student came from the above average group (having the highest total math score for this group). Based on the ahievement data alone, the novice and expert did not represent student with the lowest or highest standardized test scores, but did represent the lowest and highest overall performance scores for this study. In discussing their performance it is clear that differences in their understanding of fraction, accessiblity of fraction knowledge, tactical and strategic knowledge all lead these students to very different performance and degrees of success. In reference to the standardized test scores, both the novice and expert had identical scores of 91% for computations, but very different scores for concepts

and applications. A score of 46% for subject 061 and 92% for subject 132. This disparity in scores is congruent with the subjects' performance and competence displayed in this study.

Discussion

Lesh et al. (1983) define a conceptual model as an adaptive structure containing: (a) with-in concept networks, (b) between concept systems, (c) representational systems, and (d) modeling mechanisms. These components will be illustrated for a part-whole fraction conceptual model. Using this framework, the responses of the novice and expert fraction students will be used to speculate about their respective fraction conceptual models.

With-in Concept Networks

For Lesh et al., with-in concept networks consist of relations and operations coordinated to make judgements about the concept. For a fraction conceptual model, there could exist with-in concept networks for each type of fraction (e.g. unit versus non-unit). For example, subjects may be able to generate equivalent fractions and recognize equivalent fractions for 1/2 or 3/4. Or they might recognize that fractions can be equal to whole numbers, as when two halves equal one whole. Each of these examples are taken from the expert fraction students interview, whereas there does not exist evidence for with-in concept networks for the novice.

Between Concept Systems

Between concept systems are seen to link and/or combine with-in concept networks. For a fraction conceptual model links could be established with other rational number subconcepts (e.g. ratios,

decimals) or other conceptual systems (e.g. monetary systems as in Lawler's (1981) discussion of micro-worlds). Subject 132 seems to be establishing such links as in the fraction story about the relative cost of a dozen and half dozen eggs. I see no such links on the part of the novice, subject 061.

Representational Systems

Representational systems provide qualitatively different ways to express understanding of a concept. These include symbols, words, pictures and manipulatives. The expert fraction student seems to be equally adept at using each of these representational forms. She can use the different models to show her understanding of fraction and relates this understanding in a consistent way to the symbols used. The novice is unstable in her ability to represent fractions and in her use of the symbol as well. Neither student has a mature concept of fraction and its applications, as can be seen from their number line performance, but, the expert clearly has a flexibility that does not exist for the novice. This enables her to solve fraction problems in novel ways mapping from one representation to another (as in the 3/4 of 6 example).

Modeling Mechanisms

Modeling mechanisms are processes for: (a) changing a real situation to fit existing understanding (like Piaget's notion of assimilation), (b) changing understanding to fit the situation (accomodation for Piaget), (c) changing a model to fill gaps, eliminate inconsistencies, resolve conflicts. They enable an individual to develop and adapt based on everyday experience. For this study, subjects might relate their understanding of fraction to problems they had encountered, or

might encounter in the future, so that their conceptual model of fraction is enhanced by experience or their experience might be interpreted using their conceptual model. Subject 132 clearly uses a part-whole geometric modeling mechanism to solve the problem "3/4 of 6 disks". Hopefully the conceptual model of the expert and novice here will continue to develop in the course of their formal instruction in school and informal experience outside of the classroom.

References

Behr, M.J., Lesh, R., Post, T.R., & Silver, E. (1983). Rational-number concepts. In R. Lesh & M. Landau (Eds.), Acquistion of mathematics concepts and processes. New York: Academic Press.

Behr, M. J., Post, T. R., & Wachsmuth, I. (1982, March). Children's acquisition of order and equivalence concepts for rational numbers: Report on a clinical teaching experiment. Paper presented at the annual meeting of the American Educational Research Association, New York City.

Brown, J. S., & Burton, R. B. (1978). Diagnostic models for procedural bugs in basic mathematics skills. Cognitive Science, 2, 155-192.

Carpenter, T. M., Corbitt, M. K., Kepner, H. S., Lindquist, M. M. & Reys, R. E. (1981). National Assessment. In E. Fennema (Ed.) Mathematics education research: Implications for the 80's. (pp. 22-38). Alexandria, VA: Association for Supervision and Curriculum Development.

Carpenter, T. P., Mathews, W., Lindquist, M. M., & Silver, E. A. (1984). Achievement in mathematics: results from the national assessment. Elementary School Journal, 84(5), 485-495.

Cobb, P. (1984). The importance of beliefs and expectations in the problem solving performance of second grade pupils. In J. M. Moser (Ed.) Proceedings of the sixth annual meeting of PME-NA North American chapter of the International Group for the Psychology of Mathematics Education (pp. 135-140).

Erlwanger, S. H. (1975). Case studies of children's conceptions of mathmatics- Part I. Journal of Children's Mathematical Behavior,

1, 157-283.

Greeno, J.G. (1983). Forms of understanding in mathematical problem solving. In S.G. Paris, G.M. Olson, & H.W. Stevenson (Eds.), Learning and motivation in the classroom. Hillsdale, NJ: Lawrence Erlbaum Associates.

Hiebert, J. & LeFevre, P. (in press). Conceptual and procedural knowledge in mathematics an introductory analysis.

Hiebert, J. (1984, May). Children's learning: The struggle to link form and understanding. Elementary School Journal, 84(5).

Hiebert, J. & Tonnessen, L. H. (1978, November). Development of the fraction concept in two physical contexts: An exploratory investigation. Journal of Research in Mathematics Education, 9(5), 374-378.

Kieren, T. E. (1976). On the mathematical, cognitive and instructional foundations of rational numbers. In R. Lesh (Ed.), Number and measurement: Papers from a research workshop (pp. 101-144). Columbus, OH: ERIC/SMEC.

Larkin, J.H. (1983a). The role of problem representation in physics. In D. Gentner & A.L. Stevens (Eds.), Mental models. Hillsdale, NJ: Lawrence Erlbaum Associates.

Larkin, J.H. (1983b). Working towards a unified theory of how people use mathematics. Paper presented at the annual meeting of the American Educational Research Association, Montreal, Canada.

Larson, C. N. (1980). Seventh-grade students' ability to associate proper fractions with points on the number line. In T. E. Kieren (Ed.), Recent research on number learning (pp. 151-166).

Columbus, OH: ERIC/SMEC.

Lawler, R.W. (1981) The progressive construction of mind. Cognitive Science, 5, 1-30.

Leinhardt, G. (1983, April). Student cognitions during instruction. Paper presented at the annual meeting of the American Educational Research Association, Montreal, Canada.

Lesh, R., Landau, M., & Hamilton, E. (1983). Conceptual models and applied mathematical problem-solving research. In R. Lesh & M. Landau (Eds.), Acquisition of mathematics concepts and processes. New York: Acadmic Press.

National Assessment of Educational Progress. (1983, April). The third national mathematics assessment: Results, trends and issues. Denver,CO: Author.

Piaget, J., Inhelder, B. & Szeminska, A. (1960) The child's concept of geometry. New York: Basic Books.

Post, T. R. (1981, May). Fractions: Results and implications from National Assessment. Arithmetic Teacher, 28, 26-31.

Post, T. F., Wachsmuth, I., Lesh, R., & Behr, M. (1985). Order and equivalence of rational numbers: a cognitive analysis. Journal of Research in Mathematics Education, 16(1), 18-36.

Pothier, Y. & Sawada, D. (1983, November). Partitioning: The emergence of rational number ideas in young children. Journal of Research in Mathematics Education, 14(4), 307-317.

Riley, M.S., Greeno, J.G., & Heller, J.I. (1983). Development of children's problem-solving ability in arithmetic. In H.P. Ginsburg (Ed.), The development of mathematical thinking.

New York: Academic Press.

Schoenfeld, A.H. (1983). Episodes and executive decisions in mathematical problem-solving. In R. Lesh & M. Landau (Eds.), Acquisition of mathematics concepts and processes. New York: Academic Press.

Siegler, R.S. & Shrager, J. (1983). Strategy choices in addition: How do children know what to do? In C. Sophian (Ed.), Origins of cognitive skills.

Simon, H.A. (1978). Information-processing theory of human problem solving. In W.K. Estes (Ed.), Handbook of learning and cognitive processes. Vol.5. Human information processing. Hillsdale, NJ: Lawrence Erlbaum Associates.

Strauss, S. & Stavy, R. (1982). U-shaped behavioral growth: Implications for theories of developement. In W.W. Hartup, I.M. Ahammer & H.L. Pick (Eds.), Review of child development research. Chicago: University of Chicago Press.

VanEngen, H. (1949) An analysis of meaning in arithmetic. I. Elementary School Journal, 321-329, 395-400.

Van Lehn, K. (1983). On the representation of procedures in repair theory. In H. P. Ginsburg (Ed.), The development of mathematical thinking. (pp. 197-252). New York: Academic Press.

Vergnaud, G. (1983). Multiplicative structures. In R. Lesh & M. Landau (Eds.), Acquistion of mathematics concepts and processes. New York: Academic Press.

Appendix A

Fraction Interview

Form A

A. Fraction Concepts, Beliefs and Perceived Utility

1. Fraction Definition

---What is a fraction? (Paper and pencil and concrete materials will be available.)

---Are there some examples you could give? (If subject has difficulty, ask them about sharing experiences and whether these could be thought of using the idea of fractions.)

2. Utility of Fractions Within the Mathematics Curriculum

---Why are we learning about fractions?

---When do you think we'll study fractions again in math?

--this year?

--in high school?

---Will fractions look the same when they are studied in high school?

---What do you think we'll be doing with fractions in college?

3. Utility of Fractions in Other School Contexts

---What are some ways that fractions are used in other classes? Which classes?

Probe--Science? --Social Studies?

4. Utility of Fraction Outside of School

---Think of the adults you know and the kind of work they do. Which people do you think use fractions?

Probe--cooks? --firemen? --doctors?

---How would they use them?

---Can you think of a story where fractions might be used?

---Have you used them outside of school? When?

5. Comparison of Fractions to Whole Numbers

---How are fractions different from the other numbers we've studied, like 15 or 125?

---How are they the same?

---Are fractions easier, about the same, or more difficult to learn about than other numbers?

B. Syntax of Fractions Related to Semantics

1. Fraction Symbol

——How do you write fractions?

——Is there another way to write the same amount as 1/2?

—Probe- Does 1/2 = 2/1? Why?

——What would it mean if you wrote just one of the numbers, e.g. 1/ or /2?

2. Referents for the Fraction Symbol

——What do the top and bottom numbers in the symbol mean?

——What does the bar (line) mean?

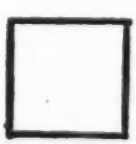
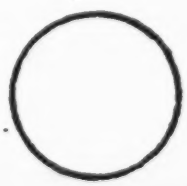
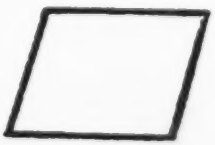
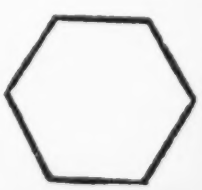
——Could you use another math symbol in place of the line to mean the same thing? (If no response, ask if ÷ could be used instead of the line, for example, 1/2 is like 1 ÷ 2.)

C. Fraction Understanding

1. Area Representations

——Can you use the shapes and straws to show me what:

—1/2 would be?

—1/3 would be?

—3/4 would be?

——Can you use this sheet of paper to fold it and label the parts as:

—1/2? yes no

—1/3? yes no

—3/4? yes no

2. Discrete Set Representations

——If I give you some counters (plastic disks), can you show me an example of:

— 1/2? (Use 2, 10, and 5)

2 10 5

— 1/3? (Use 3, 12, 7)

3 12 7

— 3/4? (Use 4, 12, 6)

4 12 6

3. Linear Representations

---Could you use this string to show these fractions:

-- 1/2? yes no

-- 1/3? yes no

-- 3/4? yes no

D. Metacognitive Knowledge about Fraction Learning

1. Misconceptions about Fractions

---If a student was confused about what a fractions is what kind of mistakes would they make with fractions, for instance, in doing the problem 1/2 + 1/4?

---What wrong idea might they have about the way fractions are written?

---Why is this wrong?

2. Acquisition of the Fraction Concept

---Which is the easiest, the most difficult to use?

--pictures of area models of fractions?

--paper-folding models?

--set-subset models?

--number lines?

*adapted from Leinhardt, 1983 and Behr et al. 1983.

Appendix A

Fraction Interview

Form B

A. Fraction Concepts, Beliefs and Perceived Utility

1. Fraction Definition

---What is a fraction? (Paper and pencil and concrete materials will be available.)

---Are there some examples you could give? (If subject has difficulty, ask them about sharing experiences and whether these could be thought of using the idea of fractions.)

2. Utility of Fractions Within the Mathematics Curriculum

---How will it help us to learn mathematics if we know about fractions?

---When do you think we'll study fractions again in math?

--next year?

--in college?

---Will fractions look the same when they are studied in college?

---What do you think we'll be doing with fractions in college?

3. Utility of Fractions in Other School Contexts

---What are some ways that fractions are used in other classes? Which classes?

Probe--Science? --Social Studies?

4. Utility of Fraction Outside of School

---Think of the adults you know and the kind of work they do. Which people do you think use fractions?

Probe--cooks? --firemen? --doctors?

---How would they use them?

---Can you think of a story where fractions might be used?

---Can you think of any time when you might use fractions? How?

---How do you think you will use fractions when you get out of school?

5. Comparison of Fractions to Whole Numbers

---How are fractions different from the other numbers we've studied, like 15 or 125?

---How are they the same?

—Are fractions easier, about the same, or more difficult to learn about than other numbers?

B. Syntax of Fractions Related to Semantics

1. Fraction Symbol

—How do you write fractions?

—Is there another way to write the same amount as 1/2?

—Probe- Does 1/2 = 2/1? Why?

—Why are there two numbers in the symbol?

2. Referents for the Fraction Symbol

—What do the top and bottom numbers in the symbol mean?

—What does the bar (line) mean?

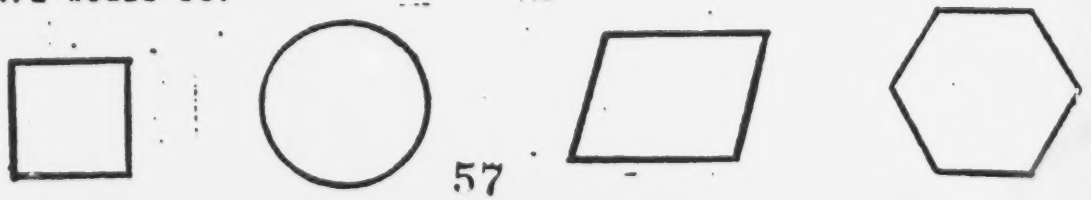
—Could you use another math symbol in place of the line to mean the same thing? (If no response, ask if ÷ could be used instead of the line, for example, 1/2 is like 1 ÷ 2.)

C. Fraction Understanding

1. Area Representations

—Can you use the shapes and straws to show me what:

—1/2 would be?



--1/3 would be?

--3/4 would be?

——Can you use this sheet of paper to fold it and label the parts as:

—1/2?

—1/3?

—3/4?

2. Discrete Set Representations

——If I give you some counters (plastic disks), can you show me an example of:

— 1/2? (Use 2, 10, and 5)

2 10 5

— 1/3? (Use 3, 12, 7)

3 12 7

— 3/4? (Use 4, 12, 6)

4 12 6

3. Linear Representations

---Could you use this string to show these fractions:

-- 1/2?

-- 1/3?

-- 3/4?

D. Metacognitive Knowledge about Fraction Learning

1. Misconceptions about Fractions

---If a student was confused about what a fractions is what kind of mistakes would they make with fractions, for instance, in doing the problem 1/2 + 1/4?

---What wrong idea might they have about the way fractions are written?

---Why is this wrong?

2. Acquisition of the Fraction Concept

---Which helps the most, the least, in trying to understand what a fraction is? (Pictures of each will be available.)

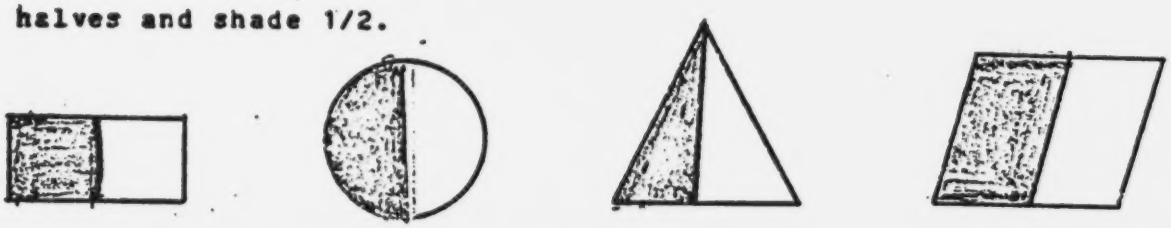
--area (region) models of fractions?

--paper-folding models?

--set-subset models?

--number lines?

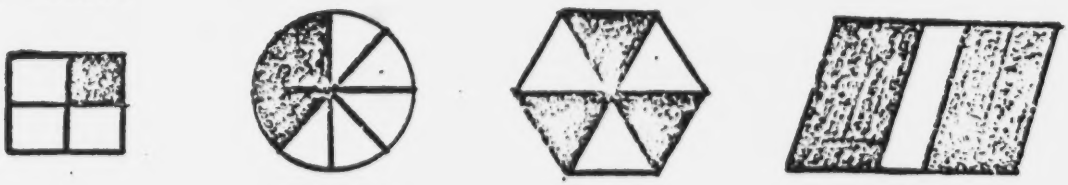
1. Draw a line (or lines) which separates each figure into halves and shade 1/2.



2. Draw a line (or lines) which separates each figure into thirds and shade 1/3.

3. Draw a line (or lines) which separates each figure into fourths and shade 3/4.

4. Write the fraction that matches the shaded parts of each shape below:



5. Shade the circles in each set below to show 1/2 of the set.

a.

b.

c.

6. Shade the circles in each set below to show 1/3 of the set.

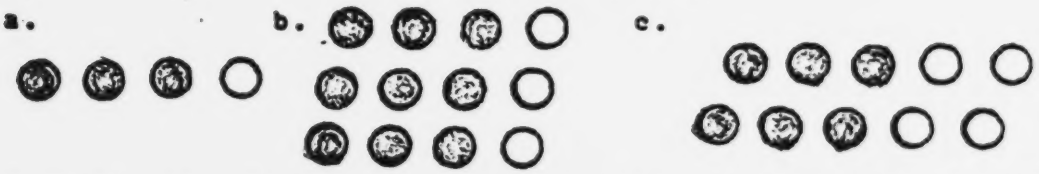
a.

b.

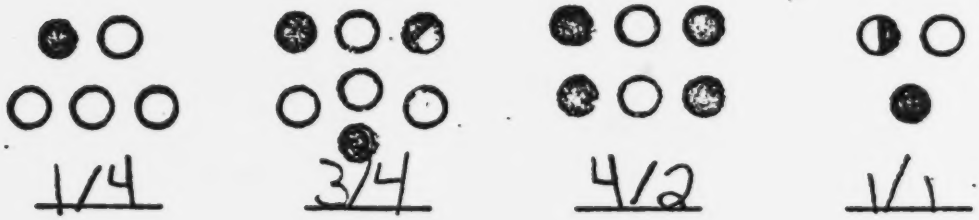
c.

7. Shade the circles in each set below to show 3/4 of the set.

a. b. c.



8. Write the fraction that matches the shaded part of each set.

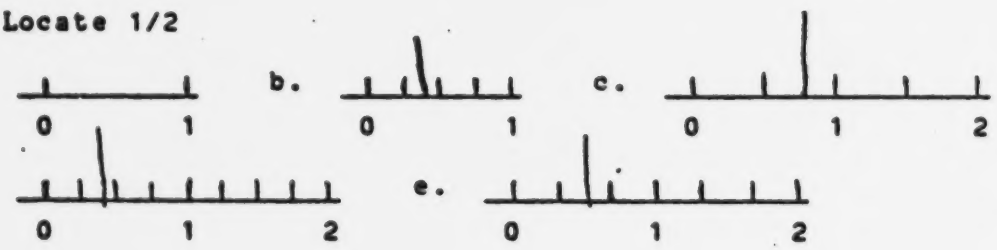


Draw a line to indicate where each of these fractions are located on the number lines below:

9. Locate 1/2

a. b. c.

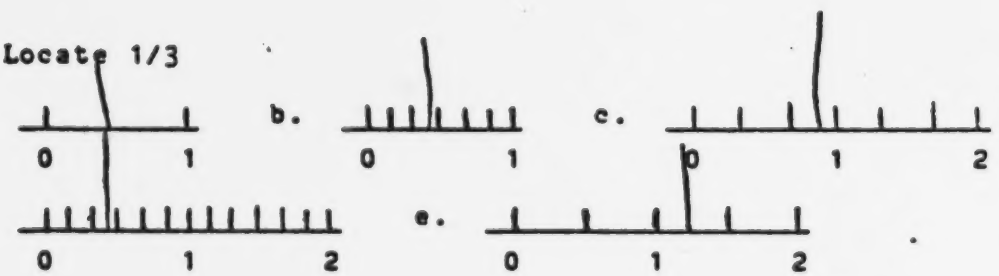
d. e.



10. Locate 1/3

a. b. c.

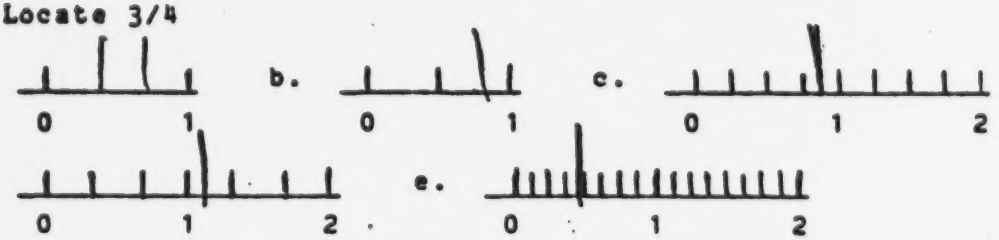
d. e.



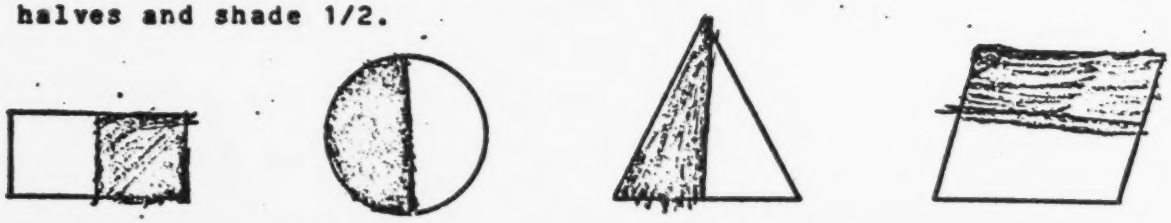
11. Locate 3/4

a. b. c.

d. e.

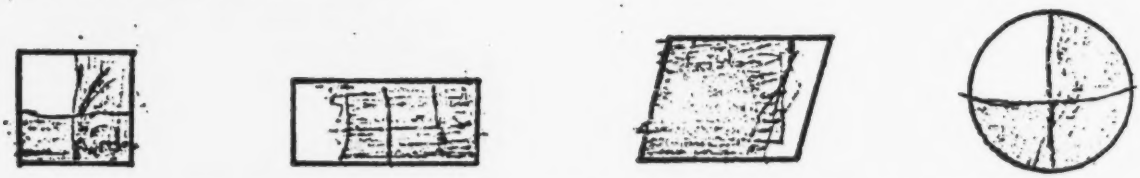


1. Draw a line (or lines) which separates each figure into halves and shade 1/2.

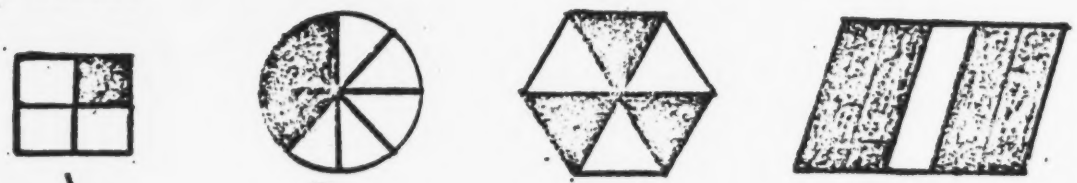


2. Draw a line (or lines) which separates each figure into thirds and shade 1/3.

3. Draw a line (or lines) which separates each figure into fourths and shade 3/4.



4. Write the fraction that matches the shaded parts of each shape below:



5. Shade the circles in each set below to show 1/2 of the set.

a.

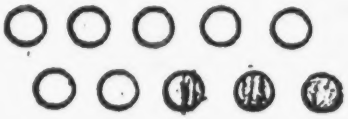
b.

c.

6. Shade the circles in each set below to show 1/3 of the set.

a.

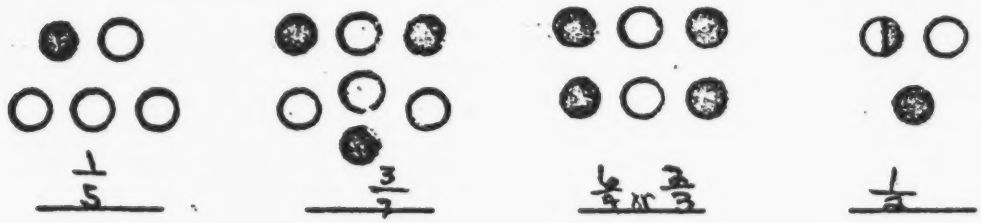
b.

c. 

7. Shade the circles in each set below to show 3/4 of the set.

a. b. c.

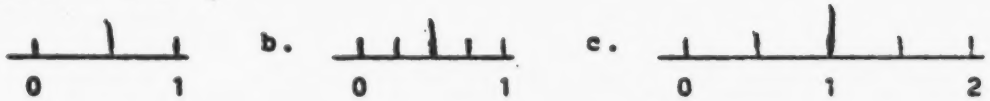
8. Write the fraction that matches the shaded part of each set.



Draw a line to indicate where each of these fractions are located on the number lines below:

9. Locate 1/2

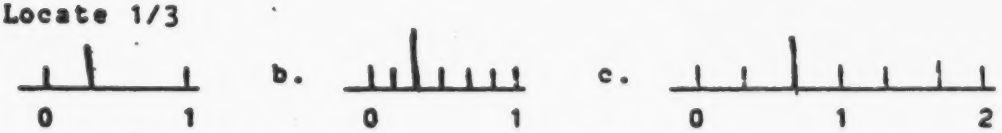
a. b. c.



d. e.

10. Locate 1/3

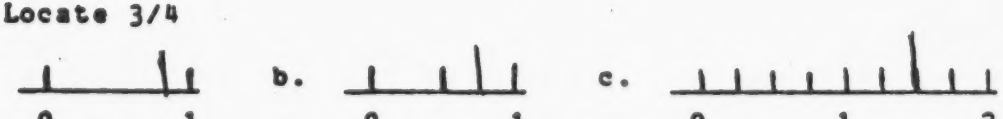
a. b. c.



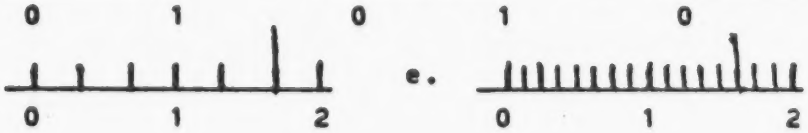
d. e.

11. Locate 3/4

a. b. c.



d. e.



Interview Data Subject 061

A. Fraction Concepts, Beliefs and Perceived Utility

1. Fraction Definition

---What is a fraction? (Paper and pencil and concrete materials will be available.)

061 no reponse

---Are there some examples you could give?

061 no response (E- Sharing a pie or pizza?) Yes ((examples because you cut them into sections))

2. Utility of Fractions Within the Mathematics Curriculum

---How will it help us to learn mathematics if we know about fractions?

061 Yes, it will help.

---When do you think we'll study fractions again in math?

	--next year?	--in college?
061-	yes	yes

---Will fractions look the same when they are studied in college?

061 look different, bigger numbers, 1000 or 1000000 in fraction

---What do you think we'll be doing with fractions in

college?

061 (not asked)

3. Utility of Fractions in Other School Contexts

---What are some ways that fractions are used in other classes? Which classes?

061 not sure

4. Utility of Fraction Outside of School

---Think of the adults you know and the kind of work they do. Which people do you think use fractions?

061 Respiratory Therapists-mom and dad

---How would they use them?

061 figure out how much medicine

---Can you think of a story where fractions might be used?

061 no

---Can you think of any time when you might use fractions? How?

061 playing a game...parchessi...figure out the number of pieces-divide up equally when some are missing

---How do you think you will use fractions when you get out of school?

061 might use them

5. Comparison of Fractions to Whole Numbers

---How are fractions different from the other numbers we've studied, like 15 or 125?

061 they're lower than 15

---How are they the same?

061 no response

---Are fractions easier, about the same, or more difficult to learn about than other numbers?

	easier	about the same	more difficult
061			X

B. Syntax of Fractions Related to Semantics

1. Fraction Symbol

---How do you write fractions?

061 4/3 or 4

---Is there another way to write the same amount as 1/2?

061 not sure

--Probe- Does 1/2 = 2/1? Why?

061 same, yes

---Why are there two numbers in the symbol?

061 don't know

2. Referents for the Fraction Symbol

——What do the top and bottom numbers in the symbol mean?

	bottom	top
061	no	no

——What does the bar (line) mean?

061 divides them (separates)

Could ÷ be used instead of the line, for example,

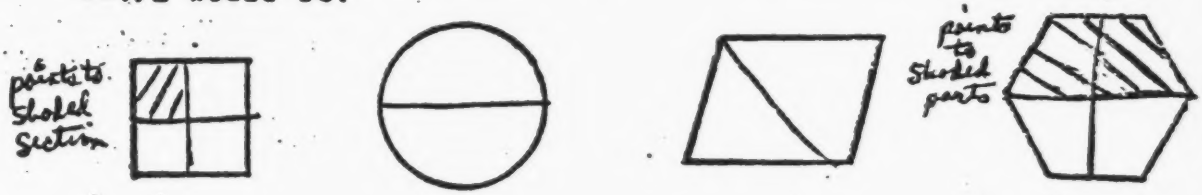
is 1/2 like 1 ÷ 2?

061 maybe-don't mean the same thing...could use another symbol for 1/2.

1. Area Representations

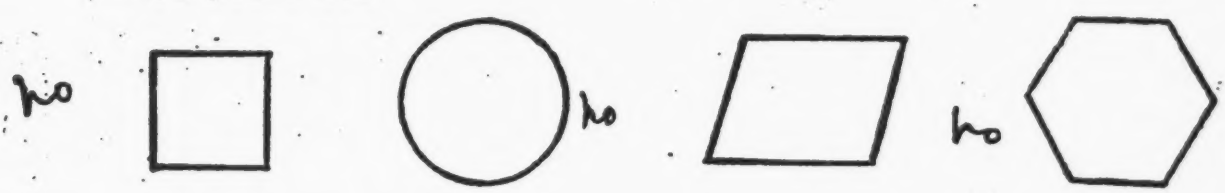
——Can you use the shapes and pipe cleaners to show me what:

——1/2 would be?



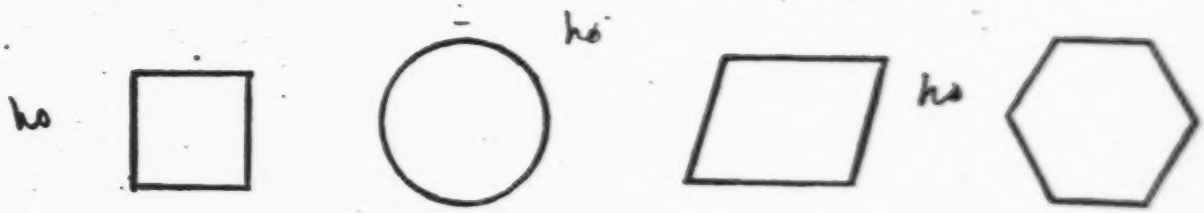
——Can you use the shapes and pipe cleaners to show me what:

——1/3 would be?



——Can you use the shapes and pipe cleaners to show me what:

--3/4 would be?



——Can you use this sheet of paper to fold it and label the parts as:

—1/2?

061 yes

——Can you use this sheet of paper to fold it and label the parts as:

—1/3?

061 (3 parts ≟)

——Can you use this sheet of paper to fold it and label the parts as:

—3/4?

061 (4 parts, 3 ≟, 1 ≠)

2. Discrete Set Representations

——If I give you some counters (plastic disks), can you show me an example of:

— 1/2? (Use 2, 10, and 5)

(2) (10) (5)

061 1 5 2 each and a half each (not sure about the question

at first)

—If I give you some counters (plastic disks), can you show me an example of:

— 1/3? (Use 3, 12, and 7)

(3) (12) (7)

061 no response

—If I give you some counters (plastic disks), can you show me an example of:

— 3/4? (Use 4, 12, and 6)

(4) (12) (6)

061 3 don't know start with 3 out of 4, 2 left, can't solve.

3. Linear Representations

—Could you use this string to show these fractions:

— 1/2?

061 yes

—Could you use this string to show these fractions:

— 1/3?

061 no 1 1/2: 6 1/2 in. made three folds

—Could you use this string to show these fractions:

— 3/4?

061 no 4 3/4: 8 1/8 made 4 folds

D. Metacognitive Knowledge about Fraction Learning

1. Misconceptions about Fractions

---If a student was confused about what a fractions is what kind of mistakes would they make with fractions?

061 I am confused, wrong answers

2. Acquisition of the Fraction Concept

---Which helps the most, the least, in trying to understand what a fraction is? (Pictures of each will be available.)

--area (region) models of fractions?

--paper-folding models?

--set-subset models?

--number lines?

	area	paper-folding	sets	number lines
061	most			least

Interview Data Subject 132

A. Fraction Concepts, Beliefs and Perceived Utility

1. Fraction Definition

---What is a fraction? (Paper and pencil and concrete materials will be available.)

132 A fraction is a part of a whole.

---Are there some examples you could give?

132 ... like 1/2 is half of a whole- it's split into 2 even parts (E- What does even mean?) equally.

2. Utility of Fractions Within the Mathematics Curriculum

---Why are we learning about fractions?

132 It's something kinda important in math. Because when you get older probably they'll have an uneven divisor problem and you'll have to say like 1/2 or 1/3 of this or that.

---When do you think we'll study fractions again in math?

	--this year?	--in high school?
132-	yes	yes

---Will fractions look the same when they are studied in high school?

132 probably not, because it will probably be harder. probably they'll have something like 23/25 ÷ 43/55 or something like that.

---What do you think we'll be doing with fractions in college?

132 I think, like they'd have stuff like a regular number divided by fractions or fractions divided by a regular number. Or, maybe something even harder than that.

3. Utility of Fractions in Other School Contexts

---What are some ways that fractions are used in other classes? Which classes?

132-I guess in some kinda advanced like Boilogy or something like that.

Probe--Science? --Social Studies?

132 Maybe in Science, I'm not sure about Social Studies though.

4. Utility of Fraction Outside of School

---Think of the adults you know and the kind of work they do. Which people do you think use fractions?

132 Math teacher, Scientist

---How would they use them?

132 Like 1/3 of this chemical goes into 1/4 of that chemical and stuff like that.

---Can you think of a story where fractions might be used?

132 Once upon a time there was a little girl in America. And she went to the supermarket...and they were selling eggs at 36¢

a dozen. And she only had enough money to buy 6 eggs and the money that she paid for that would be 18¢-half of the 36¢, because 6 is half of a dozen. 71

——Have you used them outside of school? When?

132 I had 2 friends over and they both couldn't eat one apple, so I had to split it in half.

5. Comparison of Fractions to Whole Numbers

——How are fractions different from the other numbers we've studied, like 15 or 125?

132 A fraction is kinda different from those numbers beacuse it's a lot smaller than those numbers and like in fractions they're different the way they're written.

——How are they the same?

132 They're kinda the same because if you say 2 halves that's equal to a whole.

——Are fractions easier, about the same, or more difficult to learn about than other numbers?

	easier	about the same	more difficult
132			X Some of them get more difficult and others aren't-like 1/2, 1/3 are less difficult and I heard my cousin was doing changing denominators, she's in the

fifth grade. And that's the harder part about it.

B. Syntax of Fractions Related to Semantics

1. Fraction Symbol

---How do you write fractions?

132 125/126

---Is there another way to write the same amount as 1/2?

132 3/6

—Probe- Does 1/2 = 2/1? Why?

132 no- because 2/1 is equal to 2 wholes and 1/2 is equal to one half of one whole.

---What would it mean if you wrote just one of the numbers, e.g. 1/ or /2?

132 no. If you left out the numerator then you wouldn't know how many of the denominator you had and if you left out the denominator the numerator would mean nothing, 'cause it could have been 1/16 or 1/8 or 1/4 or something like that.

2. Referents for the Fraction Symbol

---What do the top and bottom numbers in the symbol mean?

bottom top

132 it means that the whole is 125 of the 126 parts divided into 126 parts.

——What does the bar (line) mean?

132 I'm really not...I think so. I think it, um, means that like to separate the two numbers from each other.

Could ÷ be used instead of the line, for example, is 1/2 like 1 ÷ 2?

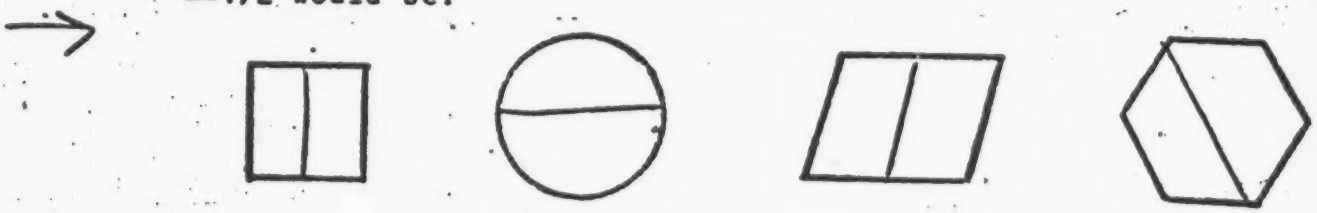
132 Um, kind of, because you'd have to split up the 1 into halves or fourths-Oh, no, it won't mean the same thing, because you have to split up 1 into fourths to make it go into 2.

C. Fraction Understanding

1. Area Representations

——Can you use the shapes and pipe cleaners to show me what:

——1/2 would be?

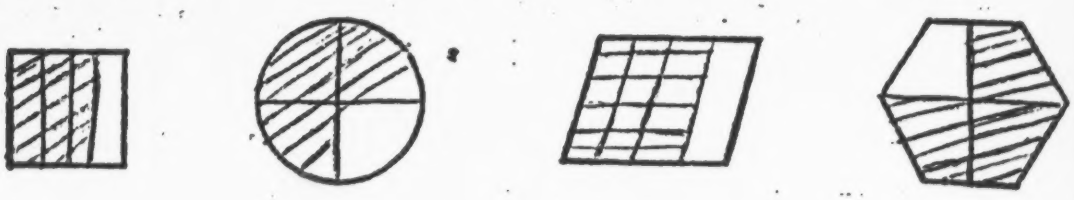


——Can you use the shapes and pipe cleaners to show me what:

——1/3 would be?

——Can you use the shapes and pipe cleaners to show me what:

——3/4 would be?



——Can you use this sheet of paper to fold it and label the parts as:

—1/2?

132 yes

——Can you use this sheet of paper to fold it and label the parts as:

—1/3?

132 yes

——Can you use this sheet of paper to fold it and label the parts as:

—3/4?

132 yes

2. Discrete Set Representations

---If I give you some counters (plastic disks), can you show me an example of:

— 1/2? (Use 2, 10, and 5)

(2) (10) (5)

132 1 5 2 my thumb is cutting half of this one

---If I give you some counters (plastic disks), can you show me an example of:

— 1/3? (Use 3, 12, and 7)

(3) (12) (7)

132 1 4 2 okay this is only cutting off 1/3 of that-pointing

---If I give you some counters (plastic disks), can you show me an example of:

— 3/4? (Use 4, 12, and 6)

(4) (12) (6)

132 3 9 4 1/2 Divide one fraction and the try to pick out how many would make...I could try dividing one little circle here into um...um 4 1/2.

(E- How did you do that?)

First I thought, how am I going to get 3/4 out of 6? And then I split some pieces up into like fourths and that didn't work...(E- That was a lot to keep track of wasn't it?) Uhuh, thirds and that didn't work.

And then you told me to move them around and then that worked. (E- Okay. How did that help you, when you moved them around?) Because, when I moved them in a circular shape, I said, 1, 2, 3, 4, and then I found that I had to split some things in half.

3. Linear Representations

---Could you use this string to show these fractions:

— 1/2?

132 yes

---Could you use this string to show these fractions:

— 1/3?

132 okay 2 1/2: 7 3/4 in.

---Could you use this string to show these fractions:

— 3/4?

132 okay 6:8

D. Metacognitive Knowledge about Fraction Learning

1. Misconceptions about Fractions

---If a student was confused about what a fractions is what kind of mistakes would they make with fractions?

132 If they get their numerators and denominators mixed up, somebody might say 3 halves in easy fractions, like when they're teaching you about halves and thirds. (E- Okay, and if they said 3 halves,

why would that be wrong?) Because it's um, 3 halves would be one [77].
whole and a half and what the person I think is trying to say is
2/3, which is one divided into 3 equal parts.

2. Acquisition of the Fraction Concept

---Which is the easiest, the most difficult to use?

--pictures of area models of fractions?

--paper-folding models?

--set-subset models?

--number lines?

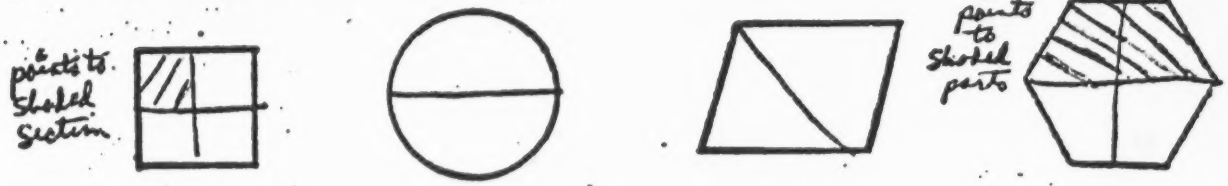
132 easy difficult Because you can get
mixed up with the number lines.

C. Fraction Understanding

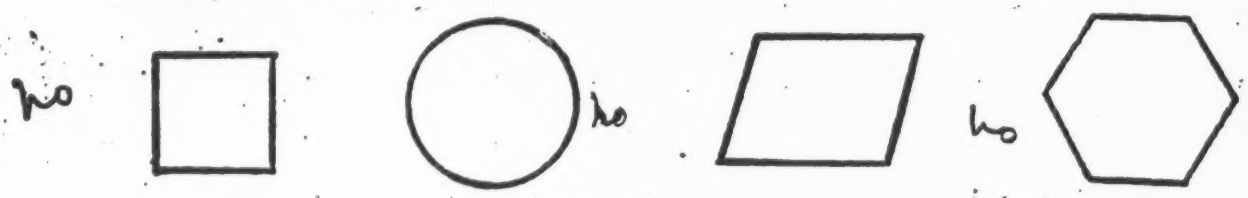
1. Area Representations

---Can you use the shapes and straws to show me what: 78 061

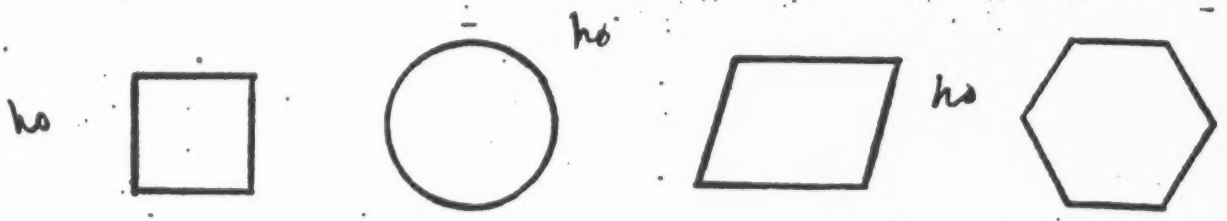
--1/2 would be?



--1/3 would be?



--3/4 would be?

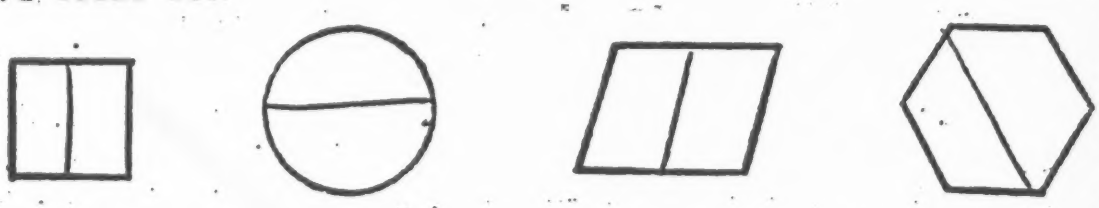


C. Fraction Understanding

1. Area Representations

---Can you use the shapes and straws to show me what: 132

--1/2 would be?



--1/3 would be?

--3/4 would be?

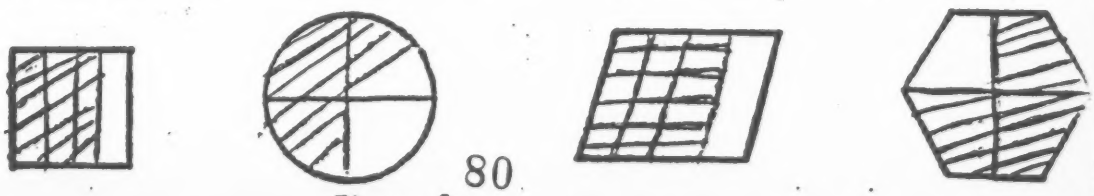


Figure 2

Appendix D

Observed Performance and Judged Ranks

011	Area Models						Linear	Discrete		Number lines			
judged	(easy) (1)									(difficult) (5)			
frac.	rec.	cir.	par.	hex.	pap.	rib.	c-c	c-m	inc	c-i	cir+	cc+	inc.
011	pretest												
1/2	1	1	1				1	1		1			
1/3								1		1			
3/4	1												
011	interview												
frac.	rec.	cir.	par.	hex.	pap.	rib.	c-c	c-m	inc	c-i	cir+	cc+	inc.
1/2	1	1	1	1	1	1	1						
1/3	1					1							
3/4					1								
total	4	2	2	1	2	2	2	2		2			// 19
ratio		9/21			2/3	2/3		4/18			2/12		// 19/54
%		43%			67%	67%		22%			17%		// 35%
rank		(3)			(1)	(1)		(4)			(5)		
judgement		(1)									(5)		

022	Area Models						Linear	Discrete		Number lines				
judged	(most) (1)													
frac.	rec.	cir.	par.				c-c	c-m	inc	c-i	cir+	cc+	inc.	
022	pretest													
1/2	1	1	1				1	1	1	1				
1/3	1						1	1		1	1*			
3/4	1	1	1				1	1						
022	interview													
frac.	rec.	cir.	par.	hex.	pap.	rib.	c-c	c-m	inc	c-i	cir+	cc+	inc.	
1/2	1	1	1	1	1	1	1	1	1					
1/3						1	1	1	1					
3/4	1	1	1	1	1	1	1	1						
total	5	4	4	2	2	3	6	6	3	2				// 37
ratio		15/21			2/3	3/3		15/18			2/12			// 37/54
%		71%			67%	100%		83%			17%			// 69%
rank		(3)			(4)	(1)		(2)			(5)			
judgement	(1)										---			

* Two lines were drawn, one being the correct response. Therefore, this response was judged incorrect.

032	Area Models						Linear	Discrete			Number lines				
	(easy) (1)										(difficult) (5)				
frac.	rec.	cir.	par.					c-c	c-m	inc	c-i	cir+	cc+	inc.	
032	pretest														
1/2	1	1	1					1			1				
1/3	1		1					1					1	1	
3/4	1		1					1							
032	interview														
frac.	rec.	cir.	par.	hex.	pap.	rib.		c-c	c-m	inc	c-i	cir+	cc+	inc.	
1/2	1	1	1	1	1	1		1	1	1					
1/3	1					1									
3/4	1		1	1	1	1		1							
total	6	2	5	2	2	3		5	1	1	1		1	1	// 30
ratio		15/21			2/3	3/3		7/18			3/12				// 30/54
%		71%			67%	100%		39%			25%				// 56%
rank		(2)			(3)	(1)		(4)			(5)				
judgement		(1)									(5)				

041	Area Models						Linear (most)	Discrete (1)		Number lines (difficult) (5)			
frac.	rec.	cir.	par.				c-c	c-m	inc	c-i	cir+	cc+	inc.
041	pretest												
1/2	1	1	1				1	1		1			
1/3	1		1				1						
3/4	1	1					1						
041	interview												
frac.	rec.	cir.	par.	hex.	pap.	rib.	c-c	c-m	inc	c-i	cir+	cc+	inc.
1/2	1	1	1	1	1	1	1	1					
1/3						1	1						
3/4	1	1	1	1	1	1	1	1					
total	5	4	4	2	2	3	6	3		1			// 30
ratio		15/21			2/3	3/3		9/18			1/12		// 30/54
%		71%			67%	100%		50%			8%		// 56%
rank		(2)			(3)	(1)		(4)			(5)		
judgement					(1)						(5)		

052	Area Models						Linear (easy) (1)	Discrete			Number lines (difficult) (5)				
frac.	rec.	cir.	par.					c-c	c-m	inc	c-i	cir+	cc+	inc.	
052	pretest														
1/2	1	1	1					1	1	1	1				
1/3	1		1					1	1		1	1			
3/4	1		1					1			1				
052	interview														
frac.	rec.	cir.	par.	hex.	pap.	rib.		c-c	c-m	inc	c-i	cir+	cc+	inc.	
1/2	1	1	1	1	1	1		1	1	1					
1/3	1		1			1		1							
3/4	1		1		1	1		1	1						
total	6	2	6	1	2	3		6	4	2	3	1			// 36
ratio		15/21			2/3	3/3		12/18			4/12				// 36/54
%		71%			67%	100%		67%			33%				// 67%
rank		(2)			(3)	(1)		(3)			(5)				
judgement					(1)						(5)				

061	Area Models						Linear	Discrete			Number lines				
	(most) (1)										(least) (5)				
frac.	rec.	cir.	par.	hex.	pap.	rib.		c-c	c-m	inc	c-i	cir+	cc+	inc.	
061	pretest														
1/2	1	1	1					1							
1/3															
3/4								1	1						
061	interview														
frac.	rec.	cir.	par.	hex.	pap.	rib.		c-c	c-m	inc	c-i	cir+	cc+	inc.	
1/2		1	1	1	1	1		1	1	1					
1/3					1										
3/4								1							
total	1	2	2	1	2	1		4	2	1					// 16
ratio		6/21			2/3	1/3		7/18			0/12				// 16/54
%		29%			67%	33%		39%			0%				// 30%
rank		(4)			(1)	(3)		(2)			(5)				
judgement		(1)									(5)				

072	Area Models (easy) (1)						Linear	Discrete		Number lines (difficult) (5)				
frac.	rec.	cir.	par.				c-c	c-m	inc	c-i	cir+	cc+	inc.	
072	pretest													
1/2	1	1	1				1			1	1*	1*	1*	
1/3	1		1					1						
3/4	1		1				1						1*	
072	interview													
frac.	rec.	cir.	par.	hex.	pap.	rib.	c-c	c-m	inc	c-i	cir+	cc+	inc.	
1/2	1	1	1	1	1	1	1	1	1					
1/3	1					1			1					
3/4	1		1		1	1	1	1	1					
total	6	2	5	1	2	3	4	3	3	1				// 30
ratio		14/21			2/3	3/3		10/18			1/12			// 30/54
%		67%			67%	100%		56%			8%			// 56%
rank		(2)			(2)	(1)		(4)			(5)			
judgement		(1)									(5)			

* Two lines were drawn, one being the correct response. Therefore, this response was judged incorrect.

081	Area Models						Linear	Discrete		Number lines				
							(most)	(1)				(least)	(5)	
frac.	rec.	cir.	par.	hex.	pap.	rib.		c-c	c-m	inc	c-i	cir+	cc+	inc.
081	pretest													
1/2	1	1	1					1	1					
1/3	1		1					1						
3/4	1	1	1					1						
081	interview													
frac.	rec.	cir.	par.	hex.	pap.	rib.		c-c	c-m	inc	c-i	cir+	cc+	inc.
1/2	1	1	1	1	1	1		1	1					
1/3						1		1						
3/4	1	1	1	1	1	1		1	1					
total	5	4	5	2	2	3		6	3					// 30
ratio		16/21			2/3	3/3		9/18			0/12			// 30/54
%		76%			67%	100%		50%			0%			// 56%
rank		(2)			(3)	(1)		(4)			(5)			
judgement						(1)					(5)			

092	Area Models						Linear		Discrete			Number lines				
	(easy) (1)						(easy) (1)					(difficult) (5)				
frac.	rec.	cir.	par.	hex.	pap.	rib.	c-c	c-m	inc	c-i	cir+	cc+	inc.			
092	pretest															
1/2	1	1	1				1	1		1	1*	1*	1*			
1/3		1					1	1		1	1*					
3/4	1	1	1				1	1		1	1*	1*				
092	interview															
frac.	rec.	cir.	par.	hex.	pap.	rib.	c-c	c-m	inc	c-i	cir+	cc+	inc.			
1/2	1	1	1	1	1	1	1	1	1							
1/3					1	1	1	1	1							
3/4	1	1	1	1	1	1	1	1								
total	4	5	4	2	3	3	6	6	2	3				// 38		
ratio		15/21			3/3	3/3		14/18			3/12			// 38/54		
%		71%			100%	100%		78%			25%			// 70%		
rank		(4)			(1)	(1)		(3)			(5)					
judgement		(1)			(1)						(5)					

* Two lines were drawn, one being the correct response. Therefore, this response was judged incorrect.

102	Area Models						Linear	Discrete		Number lines				
frac.	rec.	cir.	par.					c-c	c-m	inc	c-i	cir+	cc+	inc.
102	pretest													
1/2	1	1	1					1	1		1			
1/3	1													
3/4	1							1	1					
102	interview													
frac.	rec.	cir.	par.	hex.	pap.	rib.		c-c	c-m	inc	c-i	cir+	cc+	inc.
1/2	1	1	1	1	1	1		1	1					
1/3	1				1	1		1	1					
3/4	1							1	1					
total	6	2	2	1	2	2		5	5		1			// 26
ratio		11/21			2/3	2/3		10/18			1/12			// 26/54
%		52%			67%	67%		56%			8%			// 48%
rank		(4)			(1)	(1)		(3)			(5)			
judgement		(1)									---			

111	Area Models						Linear	Discrete		Number lines				
	(easy) (1)									(difficult) (5)				
frac.	rec.	cir.	par.	hex.	pap.	rib.	c-c	c-m	inc	c-i	cir+	cc+	inc.	
111	pretest													
1/2	1	1	1				1			1				
1/3		1					1							
3/4	1	1	1				1							
111	interview													
frac.	rec.	cir.	par.	hex.	pap.	rib.	c-c	c-m	inc	c-i	cir+	cc+	inc.	
1/2	1	1	1	1	1	1	1	1	1					
1/3		1					1	1						
3/4	1	1	1	1		1	1	1						
total	4	6	4	2	1	2	6	3	1	1				// 30
ratio		16/21			1/3	2/3		10/18			1/12			// 30/54
%		76%			33%	67%		56%			8%			// 56%
rank		(1)			(4)	(2)		(3)			(5)			
judgement		(1)									(5)			

121	Area Models						Linear	Discrete		Number lines				
	(most) (1)									(least) (5)				
frac.	rec.	cir.	par.	hex.	pap.	rib.		c-c	c-m	inc	c-i	cir+	cc+	inc.
121	pretest													
1/2	1	1	1					1	1		X			
1/3	1		1					1			X			
3/4	1		1					1	1			X		
121	interview													
frac.	rec.	cir.	par.	hex.	pap.	rib.		c-c	c-m	inc	c-i	cir+	cc+	inc.
1/2	1	1	1	1	1	1		1	1					
1/3	1							1						
3/4	1							1	1					
total	6	2	4	1	1	1		6	4		2	1		// 28
ratio		13/21			1/3	1/3		10/18			3/12			// 28/54
%		62%			33%	33%		56%			25%			// 52%
rank		(1)			(3)	(3)		(2)			(5)			
judgement		(1)									(5)			

X A different type of linear response was given here. The subject drew a line up to where the indicated fraction would be located on the number line.

132	Area Models						Linear	Discrete			Number lines				
	(easy) (1)										(difficult) (5)				
frac.	rec.	cir.	par.					c-c	c-m	inc	c-i	cir+	cc+	inc.	
132	pretest														
1/2	1	1	1					1	1	1	1				
1/3	1	1	1					1	1		1				
3/4	1	1	1					1	1	1	1				
132	interview														
frac.	rec.	cir.	par.	hex.	pap.	rib.		c-c	c-m	inc	c-i	cir+	cc+	inc.	
1/2	1	1	1	1	1	1		1	1	1					
1/3	1	1	1	1	1	1		1	1	1					
3/4	1	1	1	1	1	1		1	1	1					
total	6	6	6	3	3	3		6	6	5	3				// 47
ratio		21/21			3/3	3/3			17/18			3/12			// 47/54
%		100%			100%	100%			94%			25%			// 87%
rank		(1)			(1)	(1)			(4)			(5)			
judgement		(1)										(5)			

142	Area Models						Linear	Discrete		Number lines				
	(most) (1)									(least) (5)				
frac.	rec.	cir.	par.	hex.	pap.	rib.	c-c	c-m	inc	c-i	cir+	cc+	inc.	
142	pretest													
1/2	1	1	1				1	1		1				
1/3	1		1					1						
3/4	1	1	1				1							
142	interview													
frac.	rec.	cir.	par.	hex.	pap.	rib.	c-c	c-m	inc	c-i	cir+	cc+	inc.	
1/2	1	1	1	1	1	1	1	1						
1/3	1						1							
3/4	1	1	1	1	1	1								
total	6	4	5	2	2	2	4	3		1				// 29
ratio		17/21			2/3	2/3		7/18			1/12			// 29/54
%		81%			67%	67%		39%			8%			// 54%
rank		(1)			(2)	(2)		(4)			(5)			
judgement		(1)									(5)			

151	Area Models						Linear (easy) (1)	Discrete			Number lines (difficult) (5)			
	frac.	rec.	cir.	par.				c-c	c-m	inc	c-i	cir+	cc+	inc.
151	pretest													
	1/2	1	1	1				1	1	1	1			
	1/3	1	1	1				1	1					
	3/4	1	1	1				1	1	1	1			
151	interview													
	frac.	rec.	cir.	par.	hex.	pap.	rib.	c-c	c-m	inc	c-i	cir+	cc+	inc.
	1/2	1	1	1	1	1	1	1	1	1				
	1/3	1	1	1		1	1	1						
	3/4	1	1		1	1		1						
	total	6	6	5	2	3	2	6	4	3	2			// 39
	ratio		19/21			3/3	2/3		13/18			2/12		// 39/54
	%		90%			100%	67%		72%			17%		// 72%
	rank		(2)			(1)	(4)		(3)			(5)		
	judgement					(1)						(5)		

Kolmogorov-Smirov Test

H_0: There does not exist a difference between observed and judged ranks. N= 15 = 0.01

H_1: Not H_0

Rank	Observed Cumulative Frequency Distribution $S_n(X)$=	Cumulative Frequency Distribution under H_0 $F_0(X)$=
1	7/15	15/15
2	9/15	15/15
3	13/15	15/15
4	15/15	15/15
5	15/15	15/15

Maximum Deviation D= maximum $F_0(X)-S_n(X)$

D= 15/15 - 7/15

D= 8/15

D= $0.5\overline{3}$

Level of significance: p< 0.01

Therefore, H_0 is rejected in favor of H_1.

Figure 2

"END OF DOCUMENT"